Are Optimal Pulse Patterns Optimal?

Bounding harmonic distortion in DC/AC power converters

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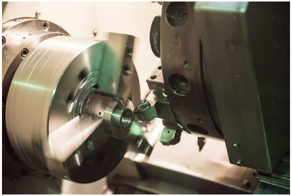
Electric Drive Systems

Motor driven by electricity

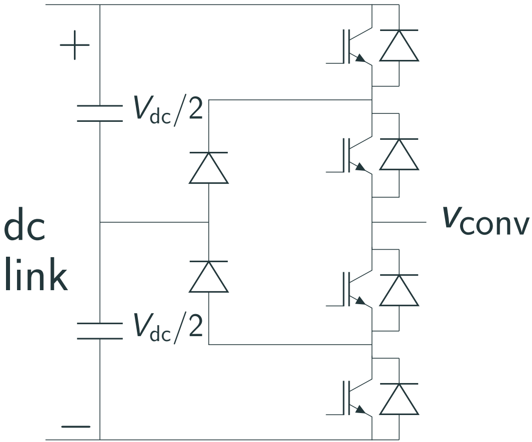


Speed/Torque/Current/Flux must be controlled

Industrial Drive Example

ABB ACS5000 DC medium voltage drive

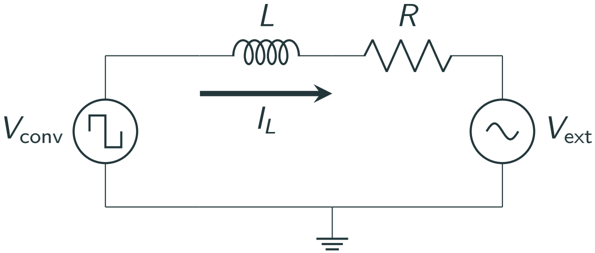
Switching Circuit



Single-phase three-level converter

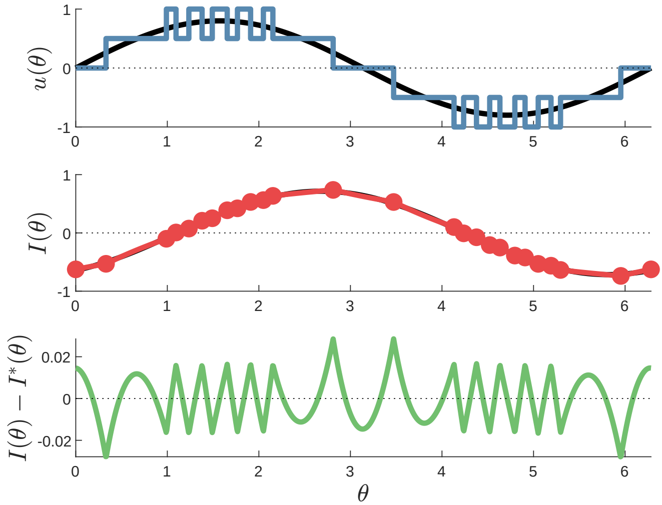
System Overview

Motor with inductance L and Resistance R



Choose V_{conv} to have smooth motor current (I_L)

Inverter Restriction, Optimal Pulse Patterns



Power quality related to $\|I(\theta)\|_2^2, \|I(\theta) - I(\theta)^*\|_2^2$

Why optimize at all?

Sources of energy inefficiency:

- Energy sent to wrong harmonics (distortion)
- Switching losses (mechanical + electrical)
- Conduction (diode and resistive)

Non-fundamental harmonics are undesirable

- Can heat and stress machines
- Increases r.m.s. current, degrades equipment lifetime

Suboptimal patterns: economic and energy losses

Main Theme

Optimal Pulse Pattern $\rightarrow$ Optimal Control

Bound control cost using convex methods

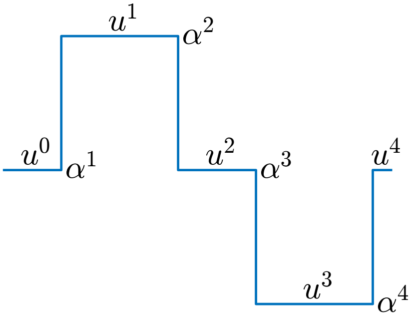
Background: Pulse Patterns

Pulse Pattern

N-level converter with levels L (e.g. $L = \{-1, 0, 1\}$, $N = 3$)

Input $u(\theta)$ described by *angles* $\{\alpha^i\}_{i=1}^k$ and *levels* $\{u^i\}_{i=0}^k$

$$u(\theta) = \begin{cases} u^0 & \theta \in [0, \alpha^1) \\ u^i & \theta \in [\alpha^i, \alpha^{i+1}) \\ u^k & \theta \in [\alpha^k, 2\pi) \end{cases}$$



Considerations for the Pulse Pattern

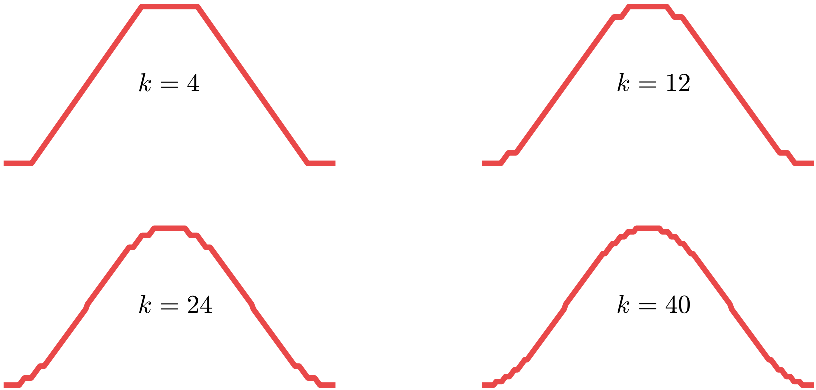
Possible constraints on design of $u(\theta)$:

- Number of switches
- Spacing
- Harmonics
- Symmetry
- Switching Restrictions
- Power Budget (Watts)

More constraints are possible (three-phase discussed later)

Constraint: Number of Switches

k switches per fundamental period (even integer)



More switches: better fidelity, more possible power losses

Constraint: Spacing

Dwell time constraint: prevents short circuits

T_s	Interlocking time	(e.g. 25 or 50 μs)
f_0	fundamental frequency	(e.g. 50 or 60 Hz)

Interlocking angle $\Theta = 2\pi f_0 T_s$ with $\forall i : \alpha_{i+1} \geq \alpha^i + \Theta$

Constraint: Harmonics

Fourier series of modulating signal:

$$u(\theta) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\theta) + b_\ell \sin(\ell\theta) \tag{1}$$

Explicit expression for Fourier coefficients in (α, u):

$$a_0 = 2u^0 + \frac{1}{\pi}\sum_{i=1}^{k}(u^{i-1} - u^i)\alpha^i \quad b_0 = 0 \tag{2}$$

$$a_\ell = \frac{1}{\ell\pi}\sum_{i=1}^{k}(u^{i-1} - u^i)\sin(\ell\alpha^i) \quad b_\ell = \frac{1}{\ell\pi}\sum_{i=1}^{k}(u^{i-1} - u^i)\cos(\ell\alpha^i)$$

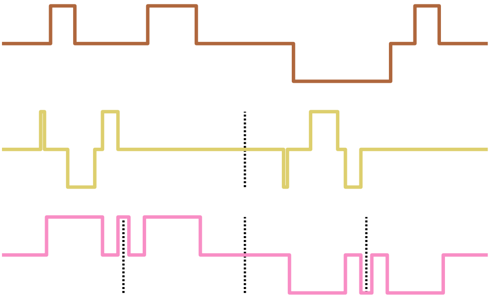
Example harmonics constraint (nonconvex in (α, u)):

$$a_0 = 0,\ a_1 = 0,\ b_1 = 1,\ a_2 \in [0, 0.1] \tag{3}$$

Constraint: Symmetries

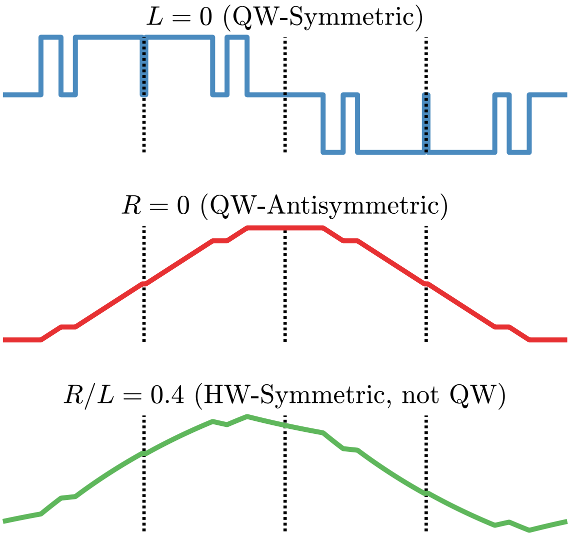
More symmetries: fewer variables, more constrained harmonics

Full-Wave	$x(\theta) = x(\theta + 2\pi)$		(4a)
Half-Wave	$x(\theta) = -x(\theta + \pi)$	$\forall\theta \in [0, \pi]$	(4b)
Quarter-Wave	$x(\theta) = x(\pi - \theta)$	$\forall\theta \in [0, \pi]$	(4c)



Constraint: Symmetries (cont.)

Resistance changes phase, no longer QW



Power Quality Objective (single-phase)

Energy in desired fundamental mode (a_1, b_1) vs. energy in I

Current Total Demand Distortion for pure reactance

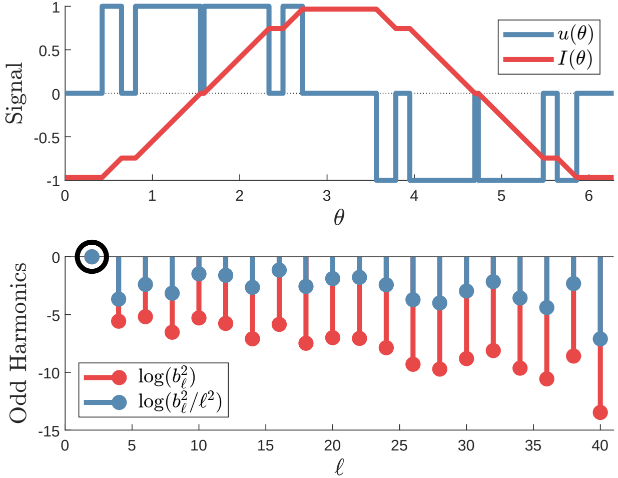
$$\text{TDD}_I \sim \sqrt{\frac{1}{\pi}\int_{\theta=0}^{2\pi} I(\theta)^2 d\theta - a_1^2 - b_1^2}$$

When a_1, b_1 are fixed (equalities)

$$\underset{u,\alpha}{\text{argmin}}\, \text{TDD}_I = \underset{u,\alpha}{\text{argmin}} \int_{\theta=0}^{2\pi} I(\theta)^2 d\theta$$

Objective: Power Quality

Energy sent to fundamental (b_1) vs. all other modes



$k = 16,\ \ b_1 = 1,$ QaHW $\implies \forall \ell :\ \ a_\ell = 0$

Background: Pattern Design

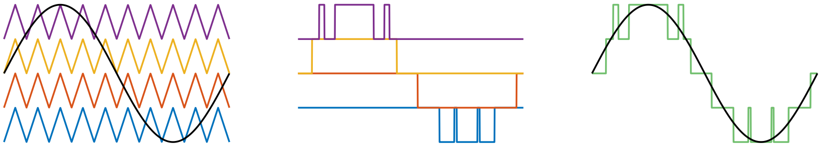
Main Approaches

Find $u(\theta)$ with minimal distortion, obeying constraints

- Carrier-Based/Space Vector Pulse Width Modulation
- Selective Harmonics *Elimination*
- Selective Harmonics *Mitigation* (e.g. IEEE519 code)
- **Optimal Pulse Patterns**
- Finite-Set Model Predictive Control

Carrier-Based Modulation

Compare reference $u^*(\theta)$ with per-level carrier $u_n^c(\theta)$



Single-phase 5-level example

Only requires comparators/lookup tables when running [1]

[1] Hava, Ahmet M., Russel J. Kerkman, and Thomas A. Lipo. "Simple analytical and graphical methods for carrier-based PWM-VSI drives." IEEE transactions on power electronics 14.1 (1999): 49-61.

Selective Harmonic Elimination

Chebyshev Polynomials can reformulate harmonics constraints

$\cos(\alpha) \to c, \quad \sin(\alpha) \to s: \quad \cos(\ell\theta) \mapsto T_\ell(c) \quad \sin(\ell\theta) \mapsto sU_{\ell-1}(c)$

Polynomial system at modulation M, choose min. TDD [2][3]

$$b_1 = M, \; b_3 = 0, \; b_5 = 0, \; b_7 = 0, \; b_9 = 0, \ldots \tag{5}$$

TDD is **not polynomial** in (c, s)

[2]Chiasson, John, et al. "A complete solution to the harmonic elimination problem." Eighteenth Annual IEEE Applied Power Electronics Conference and Exposition, 2003. APEC'03.. Vol. 1. IEEE, 2003.

[3]C. Wang, Q. Zhang, W. Yu and K. Yang, "A Comprehensive Review of Solving Selective Harmonic Elimination Problem With Algebraic Algorithms," in IEEE Transactions on Power Electronics, vol. 39, no. 1, pp. 850-868, Jan. 2024

Optimal Pulse Patterns (OPP)

Static nonconvex optimization problem[4]

$$P^* = \min_{\alpha,u,n} \mathrm{TDD}_I \tag{6a}$$

$$\text{Harmonics, Spacing Constraints} \tag{6b}$$

$$u^k = L(n^k) \tag{6c}$$

$$n^0 = n^k \text{ (periodicity)} \tag{6d}$$

$$n^{i+1} - n^i \in \{-1, 1\} \qquad \forall i \in 0..k-1 \tag{6e}$$

$$n^i \in 1..|L| \qquad \forall i \in 0..k \tag{6f}$$

SOS lower bounds via interpolating/Taylor approximation [5]

[4] Buja, Giuseppe S. "Optimum output waveforms in PWM inverters." IEEE Transactions on Industry applications 6 (1980): 830-836.

[5] Wachter, Lukas, et al. "A convex relaxation approach for the optimized pulse pattern problem." 2021 European Control Conference (ECC). IEEE, 2021.

Optimal Control Interpretation

Flow of ideas

Original OPP is static optimization

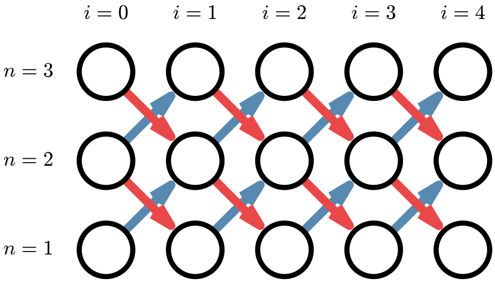
Treat OPP as a *periodic* Optimal Control Problem

Mode-selecting w.r.t. linear dynamics

Use existing techniques to bound control cost

Transition graph

Create modes: (n, i) : (voltage level, # elapsed transitions)

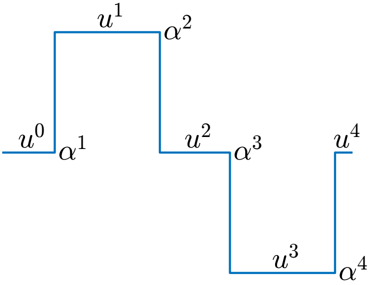


$L = \{-1, 0, 1\}, \ k = 4$

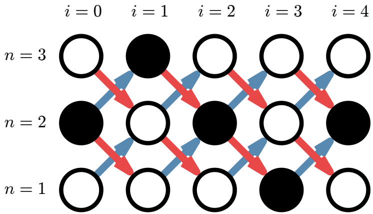
Transition graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$: $|\mathcal{V}| = N(k+1), \ |\mathcal{E}| = 2(N-1)k$

Levels to Paths

Convert levels u to path $P \in \text{Path}(\mathcal{G})$



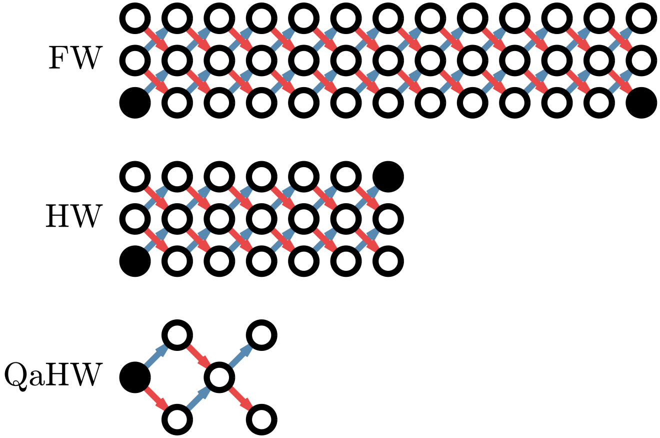
Original Pulse



Respective Path ($u \mapsto P$)

Transition Graphs under Symmetry

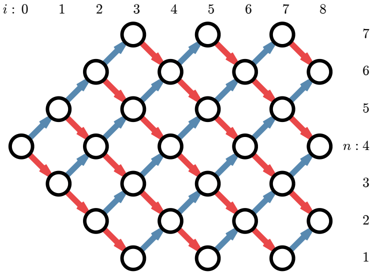
Full-Wave, Half-Wave, Quarter-and-Half Wave for $k = 12$



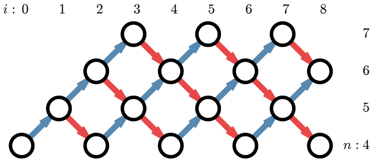
black dot: example periodicity constraint

Transition Graph: Multipolar vs. Unipolar

Unipolar: only nonnegative voltages when $\theta \in [0, \pi]$



Multipolar Transitions (QaHW)



Unipolar Transitions (QaHW)

$\mathcal{P} \subset \text{Path}(\mathcal{G})$: set of periodicity-obeying allowable paths

Per-Mode Representation

Identical states in each mode (n, i):

c	Cosine of θ
s	Sine of θ
ϕ	Angle (time) since last transition
I	Current in inductive load

Simple linear dynamics in modes, nonlinear jumps

Encoding as Hybrid Reset System

Per-mode dynamics independent of i

$$\begin{bmatrix} \text{Cosine } \theta \\ \text{Sine } \theta \\ \text{Clock Angle} \\ \text{Current} \end{bmatrix} : \quad \text{Mode } (n, i) : \quad \begin{bmatrix} \dot{c} \\ \dot{s} \\ \dot{\phi} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -s \\ c \\ 1 \\ u_n - \frac{R_{\text{load}}}{L_{\text{load}}} I \end{bmatrix} \tag{7}$$

Transition $(n, i-1) \to (n \pm 1, i)$ only if $\phi \geq \Theta$:

$$\text{Transition } (n, i, \pm) : \quad \begin{bmatrix} c_+ \\ s_+ \\ \phi_+ \\ I_+ \end{bmatrix} = \begin{bmatrix} c \\ s \\ 0 \\ I \end{bmatrix} \tag{8}$$

OCP Interpretation of OPP

OCP for TDD_I objective over curve $\mathcal{T}$:

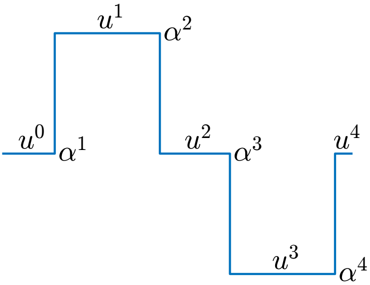
$$
\begin{aligned}
J^*_{hy} = \inf_{\phi_0, I_0, \mathcal{T}} \quad & \text{Accumulated } I^2 \text{ over } \mathcal{T} && \text{(9a)} \\
\text{s.t.} \quad & \text{Harmonics Constraints} && \text{(9b)} \\
& x(\theta) \text{ follows linear and jump dynamics} && \text{(9c)} \\
& \phi(\theta) \geq \Theta \text{ when switching} && \text{(9d)} \\
& \phi(0) = \phi_0, \ I(0) = I_0, && \text{(9e)} \\
& \text{Periodicity in state and mode} && \text{(9f)}
\end{aligned}
$$

Same optimal value as OPP, still nonconvex

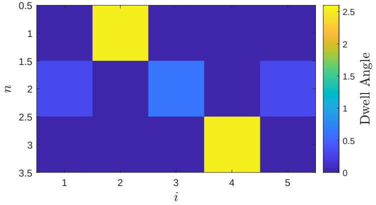
Convexification Mechanism

Dwell Table: Convexification Mechanism

Convert pattern (n, α) to mode-occupancy table



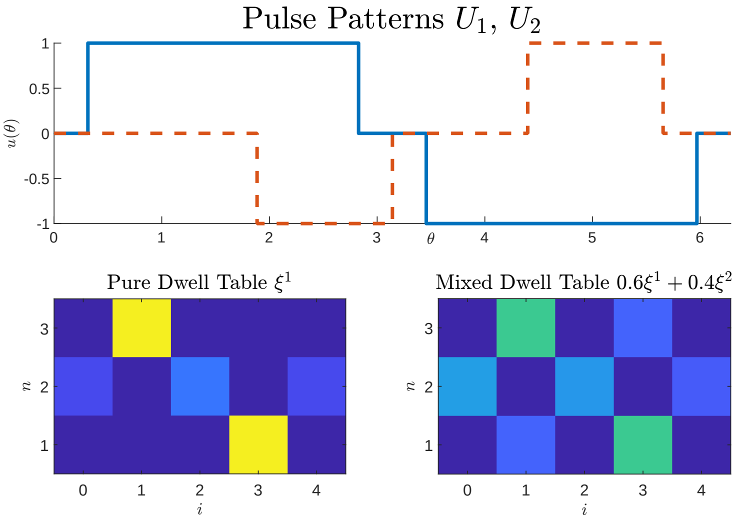
Original Pulse



Dwell/Occupancy Table

Dwell Angle $\alpha^{i+1} - \alpha^i$ is spent in mode (n, i)

Dwell Table: Convexification Mechanism



Pure and Mixed dwell table for $k = 4$

Location and Occupation (Formalization)

Location $\text{Loc}(\theta; \mathcal{T})$: mode (n, i) at angle θ

Per-mode accumulator $\Lambda_{\mathcal{T}}^{n,i}$ mapping test function $z(c, s, \phi, I)$:

$$\Lambda_{\mathcal{T}}^{n,i}[z] = \int_{\text{Loc}(\theta;\mathcal{T})=(n,i)} z(\cos(\theta), \sin(\theta), \phi(\theta), I(\theta))d\theta \quad (10)$$

$$\Lambda_{\mathcal{T}}^{n,i}[1] = \text{Time/angle spent in mode } (n, i) \quad (11)$$

Evaluation over entire trajectory

$$\Lambda_{\mathcal{T}}[z] = \sum_{(n,i)\in\mathcal{V}} \Lambda_{\mathcal{T}}^{n,i}[z] \quad (12)$$

$$\Lambda_{\mathcal{T}}[I^2] = \text{Total energy: optimal control objective} \quad (13)$$

Measure Relaxation (Occupation)

OCP is nonconvex, lift into Linear Programming in measures [6]

Introduce per-mode measure variable $\mu_{n,i}$

$$\text{Modal Occupation} \quad \mu_{n,i} \quad (n, i) \in \mathcal{V}$$

Operator $\Lambda_T^{n,i}$ is a specific instance of $\mu_{n,i}$ ($\Lambda_T^{n,i}[z] \to \int z \, d\mu_{n,i}$)

[6] Rubio, J. E. "Generalized curves and extremal points." SIAM Journal on Control 13.1 (1975): 28-47.

Measure Relaxation (Points)

Point evaluation (c, s, ϕ, l) at which event occurs

Initial	μ_n^0	$n \in L$
Terminal	μ_n^∂	$n \in L$
Step Up	$\rho_{n,i}^+$	$[(n-1, i-1) \to (n, i)] \in \mathcal{E}^+$
Step Down	$\rho_{n,i}^-$	$[(n+1, i-1) \to (n, i)] \in \mathcal{E}^-$

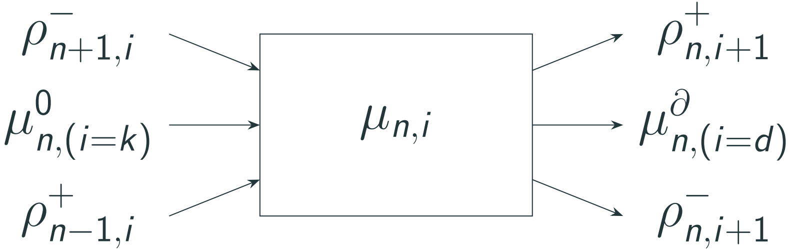
Example for initial measure derived from (α, u):

$$\mu_n^0 = \begin{cases} \delta(1, 0, \phi(0), l(0)) & u^0 = u_n \\ 0 & \text{else} \end{cases} \tag{14}$$

Interlocking: $\rho^\pm$ supported over $\phi \geq \Theta$

Continuity Relation

[Initial] + [Jump in] + [Internal Flow] = [Jump out] + [Stop]



Cost and Constraints

Cost reformulation (same process with harmonics)

$$\int_{\theta=0}^{2\pi} I(\theta)^2 d\theta \to \Lambda_{\mathcal{T}}[I^2] \to \sum_{(n,i)} \int I^2 d\mu_{n,i}(c, s, I, \phi) \qquad (15)$$

Further linear constraints on measures:

1. $\sum_n \mu_n^0$ is an initial distribution
2. μ_n^∂ appropriately matches μ_n^0 (periodicity)
3. Harmonics constraints are obeyed

Produces *lower bound* on OPP/OCP cost

Computational Complexity

All problem data polynomial: Sum-of-Squares-based SDPs[7][8]

Measures $\mu, \rho^{\pm}$ have largest size (4 states c, s, ϕ, I)

Complexity upper-bound [9] in degree β : $O((|\mathcal{V}| + |\mathcal{E}|)^{1.5}\beta^{16.5})$

Subquadratic scaling in (N, k), polynomial in β

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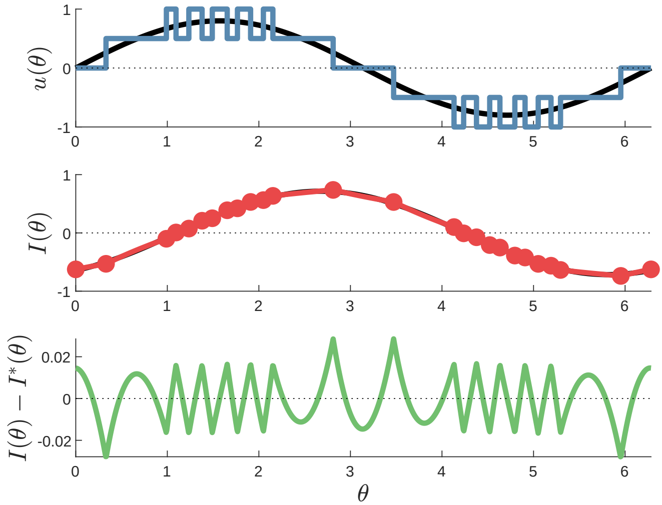
[7] Lasserre, Jean Bernard. Moments, positive polynomials and their applications. Vol. 1. World Scientific, 2009.

[8] Henrion, Didier, Jean-Bernard Lasserre, and Johan Löfberg. "GloptiPoly 3: moments, optimization and semidefinite programming." Optimization Methods & Software 24.4-5 (2009): 761-779.

[9] Claeys, Mathieu, Jamal Daafouz, and Didier Henrion. "Modal occupation measures and LMI relaxations for nonlinear switched systems control." Automatica 64 (2016): 143-154.

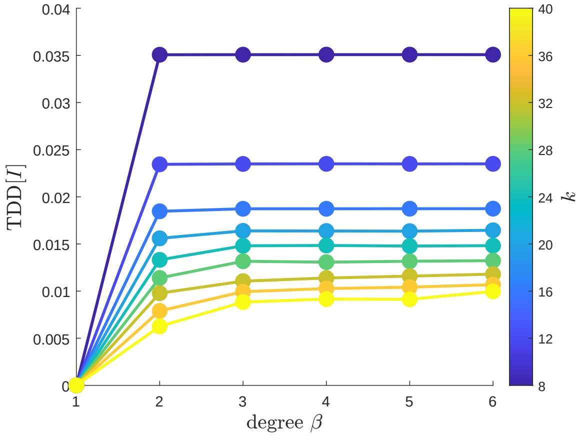
Examples

Inverter Restriction, Optimal Pulse Patterns



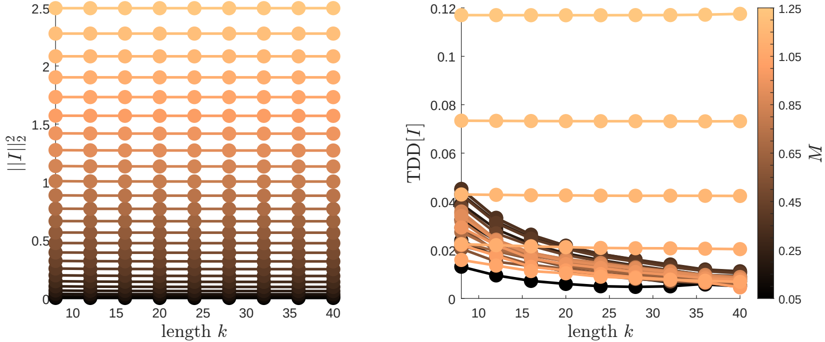
$M = 0.8,\ \|I\|_2^2 \geq 1.6092,$ TDD-suboptimality $\leq 2.2799 \times 10^{-4}$

TDD bounds vs. degree



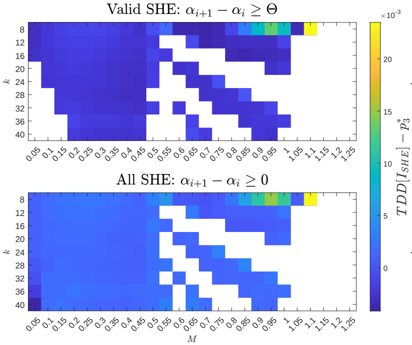
TDD bound vs. degree β and length k at $\tau = 1$

Increased number of pulses

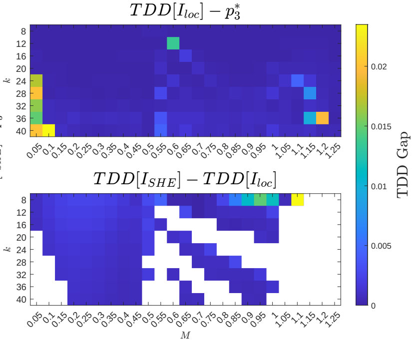


Energy and TDD vs. increasing k with $\beta = 3$

Comparison with Selective Harmonics Elimination

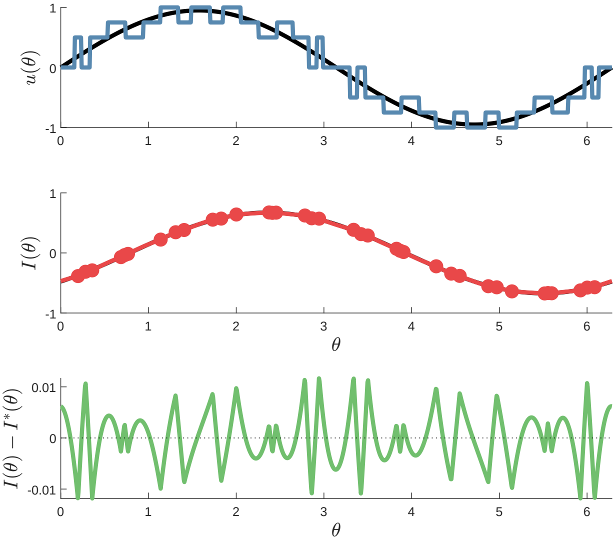


SHE solutions



SHE and $\beta = 3$ recovered

Inhomogenous Spacing



7-level inverter

Take-aways

Conclusion

Treated OPP as a dynamical systems problem

Use convex programming methods to bound optimality

(Hopefully) more efficient electric drives

Future Work

Bounded Power Losses

Three-phase differential-mode distortion

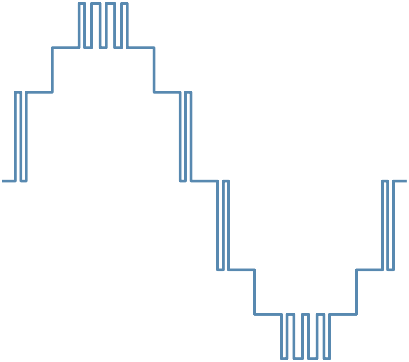
Faster, more accurate numerical computation

Experimental validation

Efficient Power!

Encore: Backup Slides

Multi-level Pulse Patterns



7-level inverter with inhomogenous spacing

Objective: Computation

Computing the objective

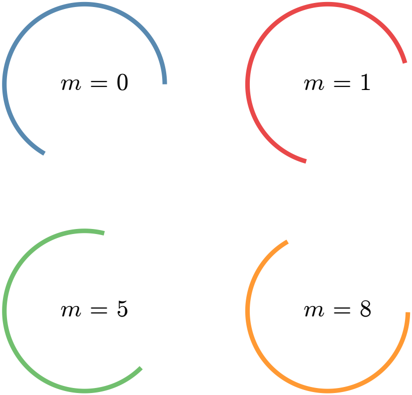
Objective	**Fixed** u	**Search over** u
TDD_V	Linear	Cubic
TDD_I	Piecewise-cubic	Piecewise-rational in
$\mathrm{TDD}_I^{3\varphi}$	Bad	Worse

$\mathrm{TDD}_I^{3\varphi}$ requires sorting α over $2\pi/3$ phase shifts

Sometimes evaluated as a finite series (e.g. $\sqrt{\sum_{\ell \neq 1,\ 3 \nmid \ell}^{500} \frac{a_\ell^2 + b_\ell^2}{n^2}}$)

Trigonometric Support

Restricted angle regions: cuts of circle ($\Delta = \pi/12$)



Must have $\geq \Theta(k - i)$ angle remaining to perform all switches

Common-Mode Voltage Constraint

u_{cm} adds stress to components

Supply constraints: $(|u_{cm}(\theta)| \leq 1/3)$ [10] or $(u_{cm}(\theta) = 0)$ [11]

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[10] Koukoula, Isavella, Petros Karamanakos, and Tobias Geyer. "Optimal pulse width modulation of three-level converters with reduced common-mode voltage." IEEE Transactions on Industry Applications 60.3 (2024): 4062-4075.

[11] Koukoula, Isavella, Petros Karamanakos, and Tobias Geyer. "Three-Level Optimized Pulse Patterns with Zero Common-Mode Voltage." 2023 IEEE 17th International Conference on Compatibility, Power Electronics and Power Engineering (CPE-POWERENG). IEEE, 2023.

Extensions

Fluctuating Voltage

DC source may fluctuate $V_{dc}(\theta)$

Inverter levels $u_n(\theta)$ also fluctuate at each n

Replace accordingly in dynamics: $\dot{I}(\theta) = u_n(\theta)$

Sources of Loss

Constrain power budget (e.g. 3000 W) [12]

Two main types of losses:

- Switching (linear penalty on $\rho^{\pm}$)
- Conduction (nonlinear penalty on $\rho^{\pm}$)
- Resistive (linear penalty on μ)

[12] Geyer, Tobias, Petros Karamanakos, and Isavella Koukoula. "Optimized pulse patterns with bounded semiconductor losses." IEEE Transactions on Power Electronics 39.3 (2023): 3233-3243.

Capacitative Load

TDD formulation developed for inductive load

Capacitor has voltage dynamics (V_C)

$$I_C = C\dot{V}_C = (V - V_C)/R \tag{16}$$

Current objective for capacitative load

$$\int_{\theta=0}^{2\pi} I_C(\theta)^2 d\theta \to \frac{1}{RC} \sum_{\ell,m} \langle V_c^2 - 2u_\ell V_C + u_\ell^2, \mu_{\ell,m} \rangle \tag{17}$$

Objective and load dynamics change, the rest remains

Grid-Side Filters

Add filters (C, LC, LCL) between voltage source and load

Design OPP w.r.t. the filter [13]

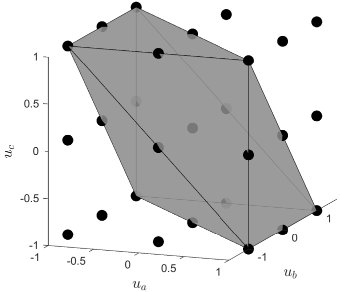
New dynamical states to track

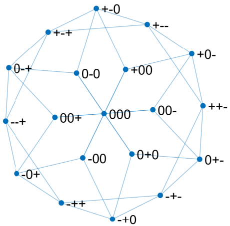
Objective, framework is the same (with more states)

[13] Rahmanpour, Shirin, Petros Karamanakos, and Tobias Geyer. "Three-level optimized pulse patterns for grid-connected converters with LCL filters." 2023 IEEE Energy Conversion Congress and Exposition (ECCE). IEEE, 2023.

Common-Mode Constraints

Only retain states with $|u_a + u_b + u_c| \leq u_{cm}$





$u_{cm} = 1/3$ modes and pruned graph

Transition graph $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$ for three-phase modes

Three-Phase

Measure Relaxation

Add new measures for Three-phase system

Initial	$\bar{\mu}^0_{\mathbf{n}}$	$\mathbf{n} \in \mathcal{V}_3$
Terminal	$\bar{\mu}^\partial_{\mathbf{n}}$	$\mathbf{n} \in \mathcal{V}_3$
Occupation	$\bar{\mu}_{\mathbf{n}}$	$\mathbf{n} \in \mathcal{V}$
Steps	$\bar{\rho}_e$	$e \in \mathcal{E}_3$

Align with single-phase μ along (c, s, I_a)

Computational Burden

Largest measure now has 5 states (c, s, I_a, I_b, I_c)

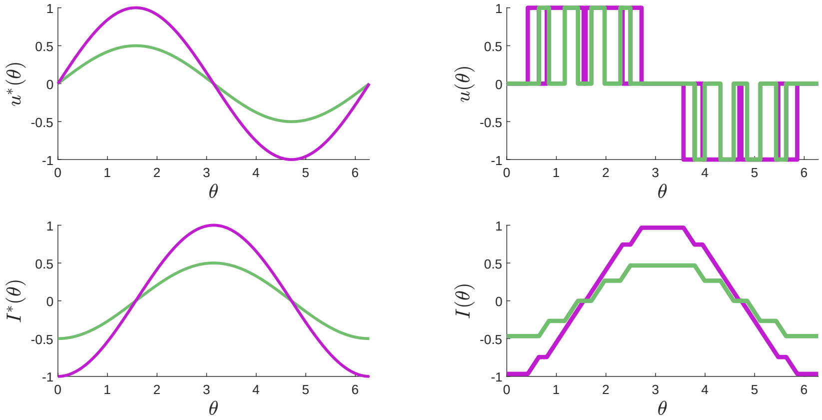
At $L = 3, |v_{cm}| \leq 1/3$: $|\mathcal{V}_3| = 19,\ |\mathcal{E}_3| = 78$

Preprocessing currently takes 5 min (at order 2)

Worse at higher degrees, 3-phase is the bottleneck

General Issue: Nonlinearity

Nonlinearity in control due to finite set of inputs



Amplitudes 0.5 and 1, nonlinear change in $u(\theta)$

Future Work (Computation)

Use intermediate relaxations (partial degree information) [14]

Try out Neural Network subvalue certificates [15]

[14] Wang, Jie. "Strengthening Lasserre's Hierarchy in Real and Complex Polynomial Optimization." arXiv preprint arXiv:2404.07125 (2024).

[15] Edwards, Alec, Andrea Peruffo, and Alessandro Abate. "Fossil 2.0: Formal certificate synthesis for the verification and control of dynamical models." Proceedings of the 27th ACM International Conference on Hybrid Systems: Computation and Control. 2024.