Bounding the Distance to Unsafe Sets with Convex Optimization

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Abstract

This work proposes an algorithm to bound the minimum distance between points on trajectories of a dynamical system and points on an unsafe set. Prior work on certifying safety of trajectories includes barrier and density methods, which do not provide a margin of proximity to the unsafe set in terms of distance. The distance estimation problem is relaxed to a Monge-Kantorovich type optimal transport problem based on existing occupation-measure methods of peak estimation. Specialized programs may be developed for polyhedral norm distances (e.g. L1 and Linfinity) and for scenarios where a shape is traveling along trajectories (e.g. rigid body motion). The distance estimation problem will be correlatively sparse when the distance objective is separable.

1 Introduction

A trajectory is safe with respect to an unsafe set X_u if no point along the trajectory contacts or enters X_u. Safety of trajectories may be quantified by the distance of closest approach to X_u, which will be positive for all safe trajectories and zero for all unsafe trajectories. The task of finding this distance of closest approach will also be referred to as 'distance estimation'. In this setting, an agent with state x is restricted to a state space $X \subseteq \mathbb{R}^n$ and starts in an initial set $X_0 \subset X$. The trajectory of an agent evolving according to locally Lipschitz dynamics $\dot{x} = f(t, x(t))$ starting at an initial condition $x_0 \in X_0$ is denoted by $x(t \mid x_0)$. The closest approach as measured by a distance function c that any trajectory takes to the unsafe set X_u in a time horizon of $t \in [0, T]$ can be found by solving,

$$\begin{aligned} P^* = \inf_{t,\, x_0, y} \ & c(x(t \mid x_0), y) \\ & \dot{x}(t) = f(t, x), \quad t \in [0, T] \\ & x(0) = x_0 \in X_0, \ y \in X_u. \end{aligned} \tag{1}$$

Solving (1) requires optimizing over all points $(t, x_0, y) \in [0, T] \times X_0 \times X_u$, which is generically a non-convex and difficult task. Upper bounds to P^* may be found by sampling points (x_0, y) and evaluating $c(x(t \mid x_0), y)$ along these sampled trajectories. Lower bounds to P^* are a universal property of all trajectories, and will satisfy $P^* > 0$ if all trajectories starting from X_0 in the time horizon $[0, T]$ are safe with respect to X_u.

This paper proposes an occupation-measure based method to compute lower bounds of P^* through a converging hierarchy of convex Semidefinite Programs (SDPs) [1]. These SDPs arise from finite truncation of infinite dimensional Linear Programs (LPs) in measures [2]. Occupation measures are Borel measures that contain information about the distribution of states evolving along trajectories of a dynamical system. The distance estimation LP formulation is based on measure LPs arising from peak estimation of dynamical systems [3, 4, 5], because the state function to be minimized along trajectories is the point-set distance function between $x \in X$ and X_u. Inspired by optimal transport theory [6, 7, 8], the distance function $c(x, y)$ between points $x \in X$ on trajectories and $y \in X_u$ is relaxed to an expectation of the distance $c(x, y)$ with respect to probability distributions over X and X_u.

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Occupation measure LPs for control problems were first formulated in [9], and their Linear Matrix Inequality (LMI) relaxations were detailed in [10]. These occupation measure methods have also been applied to region of attraction estimation and backwards reachable set maximizing control [11, 12, 13].

Prior work on verifying safety of trajectories includes Barrier functions [14, 15], Density functions [16], and Safety Margins [17]. Barrier and Density functions offer binary indications of safety/unsafety; if a Barrier/Density function exists, then all trajectories starting from X_0 are safe. Barrier/Density functions may be non-unique, and the existence of such a function does not yield a measure of closeness to the unsafe set. Safety Margins are a measure of constraint violation, and a negative safety margin verifies safety of trajectories. Safety Margins can vary with constraint reparameterization, even in the same coordinate system (e.g. multiplying all defining constraints of X_u by a positive constant scales the safety margin by that constant), and therefore yield a qualitative certificate of safety. The distance of closest approach P^* is independent of constraint reparameterization, returning quantifiable and geometrically interpretable information about safety of trajectories.

The contributions of this paper include:

- A measure LP to lower bound the distance estimation task (1)
- A proof of convergence to P^* within arbitrary accuracy as the degree of LMI approximations approaches infinity
- A decomposition of the distance estimation LP using correlative sparsity when the cost $c(x, y)$ is separable
- Extensions such as finding the distance of closest approach between a shape with evolving orientations and the unsafe set

Parts of this paper were accepted for presentation at the 61st Conference on Decision and Control [18]. Contributions of this work over and above the conference version include:

- A discussion of the scaling properties of safety margins
- An application of correlative sparsity in order to reduce the computational cost of finding distance estimates
- An extension to bounding the set-set distance between a moving shape and the unsafe set
- A presentation of a lifting framework for polyhedral norm distance functions
- A full proof of strong duality

This paper is structured as follows: Section 2 reviews preliminaries such as notation and measures for peak and safety estimation. Section 3 proposes an infinite-dimensional LP to bound the distance closest approach between points along trajectories and points on the unsafe set. Section 4 truncates the infinite-dimensional LPs into SDPs through the moment-Sum of Squares (SOS) hierarchy, and studies numerical considerations associated with these SDPs. Section 5 utilizes correlative sparsity to create SDP relaxations of distance estimation with smaller Positive Semidefinite (PSD) matrix constraints. Distance estimation problems for shapes traveling along trajectories are posed in Section 6. Examples of the distance estimation problem are presented in Section 7. Section 8 details extensions to the distance estimation problem, including uncertainty, polyhedral norm distances, and application of correlative sparsity. The paper is concluded in Section 9. Appendix A offers a proof of strong duality betwen infinite-dimensional LPs for distance estimation. Appendix B summarizes the moment-SOS hierarchy.

CSP Correlative Sparsity Pattern

LMI Linear Matrix Inequality

LP Linear Program

PSD Positive Semidefinite

SDP Semidefinite Program

SOS Sum of Squares

WSOS Weighted Sum of Squares

2 Preliminaries

2.1 Notation and Measure Theory

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be an n-dimensional real Euclidean space. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^n$ be the set of n-dimensional multi-indices. The total degree of a multi-index $\alpha \in \mathbb{N}^n$ is $|\alpha| = \sum_i \alpha_i$. A monomial $\prod_{i=1}^n x_i^{\alpha_i}$ may be expressed in multi-index notation as x^α. The set of polynomials with real coefficients is $\mathbb{R}[x]$, and polynomials $p(x) \in \mathbb{R}[x]$ may be represented as the sum over a finite index set $\mathscr{J} \subset \mathbb{N}^n$ of $p(x) = \sum_{\alpha \in \mathscr{J}} p_\alpha x^\alpha$. The set of polynomials with monomials up to degree $|\alpha| = d$ is $\mathbb{R}[x]_{\leq d}$. A metric function $c(x, y)$ over the space X with $x, y \in X$ satisfies the following properties [19]:

$$c(x,y) = c(y,x) > 0 \qquad x \neq y \tag{2a}$$

$$c(x,x) = 0 \tag{2b}$$

$$c(x,y) \leq c(x,z) + c(z,y) \qquad \forall z \in X. \tag{2c}$$

The set of metric functions are closed under addition and pointwise maximums. Every norm $\|\cdot\|$ inspires a metric $c_{\|\cdot\|}(x,y) = \|x - y\|$. The point-set distance function $c(x;Y)$ between a point $x \in X$ and a closed set $Y \subset X$ is defined by:

$$c(x;Y) = \min_{y \in Y} c(x,y). \tag{3}$$

The set of continuous functions over $X \subset \mathbb{R}^n$ is denoted as $C(X)$, the set of finite signed Borel measures over X is $\mathcal{M}(X)$, and the set of nonnegative Borel measures over X is $\mathcal{M}_+(X)$. A duality pairing exists between all functions $f \in C(X)$ and measures $\mu \in \mathcal{M}_+(X)$ by Lebegue integration: $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$ when X is compact. The subcone of nonnegative continuous functions over X is $C_+(X) \subset C(X)$, which satisfies $\langle f, \mu \rangle \geq 0 \ \forall f \in C_+(X), \ \mu \in \mathcal{M}_+(X)$. The subcone of continuous functions over X whose first k derivatives are continuous is $C^k(X)$ (with $C(X) = C^0(X)$). The indicator function of a set $A \subseteq X$ is a function $I_A : X \to \{0, 1\}$ taking values $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$. The measure of a set A with respect to $\mu \in \mathcal{M}_+(X)$ is $\mu(A) = \langle I_A(x), \mu \rangle = \int_A d\mu$. The mass of μ is $\mu(X) = \langle 1, \mu \rangle$, and μ is a probability measure if $\langle 1, \mu \rangle = 1$. The support of μ is the set of all points $x \in X$ such that every open neighborhood N_x of x has $\mu(N_x) > 0$. The Lebesgue measure λ_X over a space X is the volume measure satisfying $\langle f, \lambda_X \rangle = \int_X f(x) dx \ \forall f \in C(X)$. The Dirac delta $\delta_{x'}$ is a probability measure supported at a single point $x' \in X$, and the duality pairing of any function $f \in C(X)$ with respect to $\delta_{x'}$ is $\langle f(x), \delta_{x'} \rangle = f(x')$. A measure $\mu = \sum_{i=1}^r c_i \delta_{x_i}$ that is the conic combination (weights $c_i > 0$) of r distinct Dirac deltas is known as a rank-r atomic measure. The atoms of μ are the support points $\{x_i\}_{i=1}^r$.

Let X, Y be spaces and $\mu \in \mathcal{M}_+(X), \ \nu \in \mathcal{M}_+(Y)$ be measures. The product measure $\mu \otimes \nu$ is the unique measure such that $\forall A \in X, \ B \in Y : \ (\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$. The pushforward of a map $Q : X \to Y$ along a measure $\mu(x)$ is $Q_\# \mu(y)$, which satisfies $\forall f \in C(Y) : \ \langle f(y), Q_\# \mu(y) \rangle = \langle f(Q(x)), \mu(x) \rangle$. The measure of a set $B \in Y$ is $Q_\# \mu(Y) = \mu(Q^{-1}(Y))$. The projection map $\pi^x : X \times Y \to X$ preserves only the x-coordinate as $(x, y) \to x$ and a similar definition holds for π^y. Given a measure $\eta \in \mathcal{M}_+(X \times Y)$, the projection-pushforward $\pi^x_\# \eta$ expresses the x-marginal of η with duality pairing $\forall f \in C(X) : \langle f(x), \pi^x_\# \eta \rangle = \int_{X \times Y} f(x) d\eta(x, y)$. Every linear operator $\mathcal{L} : X \to Y$ possesses a unique adjoint operator $\mathcal{L}^\dagger$ such that $\langle \mathcal{L} f, \mu \rangle = \langle f, \mathcal{L}^\dagger \mu \rangle, \ \forall f \in C(X), \mu \in \mathcal{M}_+(X)$.

Letting $x(t) : [a, b] \to X$ be a single-valued curve parameterized by t with evaluation map $t \mapsto (t, x(t))$, we define the evaluation measure $\mu_e : \mathbf{eval}([a, b], t \mapsto (t, x(t))$ as the unique measure satisfying $\forall \phi \in C([a, b] \times X) : \langle \phi(t, x), \mu_e \rangle = \int_a^b \phi(t, x(t)) dt$. Equivalently, the evaluation measure μ_e is the pushforward of the Lebesgue distribution $\lambda_{[a,b]}$ in time along the map $t \mapsto (t, x(t)))$.

2.2 Peak Estimation and Occupation Measures

The peak estimation problem involves finding the maximum value of a state function $p(x)$ along trajectories of a dynamical system,

$$P^* = \sup_{t \in [0,T],\, x(0)=x_0 \in X_0} p(x(t \mid x_0)), \quad \dot{x}(t) = f(t, x(t)). \tag{4}$$

Every optimal trajectory of (4) (if one exists) may be described by a tuple (x_0^*, t_p^*, x_p^*) satisfying $P^* = p(x_p^*) = p(x(t_p^* \mid x_0^*))$. A persistent example throughout this paper will be the Flow system of [14]:

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 - x_2 + \frac{1}{3}x_1^3 \end{bmatrix}. \tag{5}$$

Figure 1 plots trajectories of the flow system in cyan for times $t \in [0, 5]$, starting from the initial set $X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$ in the black circle. The minimum value of x_2 along these trajectories is $\min x_2 \approx -0.5734$. The optimizing trajectory is shown in dark blue, starting at the blue circle $x_0^* \approx (1.4889, -0.3998)$ and reaching optimality at $x_p^* \approx (0.6767, -0.5734)$ in time $t_p^* \approx 1.6627$.

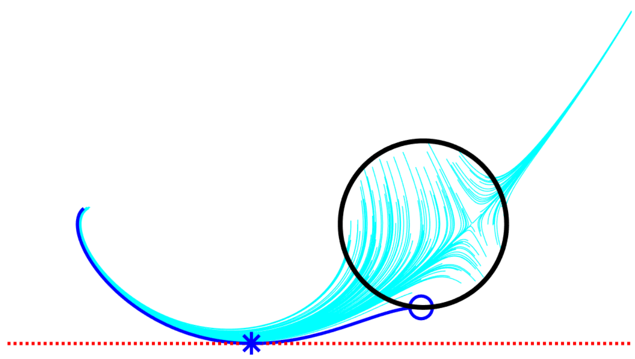


Figure 1: Minimizing x_2 along Flow system (5)

The work in [4] developed a measure LP to find an upper bound $p^* \geq P^*$. This measure LP involves an initial measure $\mu_0 \in \mathcal{M}_+(X_0)$, a peak measure $\mu_p \in \mathcal{M}_+([0,T] \times X)$, and an occupation measure $\mu \in \mathcal{M}_+([0,T] \times X)$ connecting together μ_0 and μ_p. Given a distribution of initial conditions $\mu_0 \in \mathcal{M}_+(X_0)$ and a stopping time $0 \leq t^* \leq T$, the occupation measure μ of a set $A \times B$ with $A \subseteq [0,T]$, $B \subseteq X$ is defined by,

$$\mu(A \times B) = \int_{[0,t^*] \times X_0} I_{A \times B}\left((t, x(t \mid x_0))\right) dt\, d\mu_0(x_0). \tag{6}$$

The measure $\mu(A \times B)$ is the μ_0-averaged amount of time a trajectory will dwell in the box $A \times B$. With ODE dynamics $\dot{x}(t) = f(t, x(t))$, the Lie derivative $\mathcal{L}_f$ along a test function $v \in C^1([0,T] \times X)$ is,

$$\mathcal{L}_f v(t,x) = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x). \tag{7}$$

Liouville's equation expresses the constraint that μ_0 is connected to μ_p by trajectories with dynamics f for all test functions $v \in C^1([0,T] \times X)$,

$$\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \tag{8}$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu. \tag{9}$$

Equation (9) is an equivalent short-hand expression to equation (8) for all v. Substituting in the test functions $v = 1, v = t$ to Liouville's equation returns the relations $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle$ and $\langle 1, \mu \rangle = \langle t, \mu_p \rangle$.

The measure LP corresponding to (4) with optimization variables (μ_0, μ_p, μ) is [4],

$$p^* = \sup \quad \langle p(x), \mu_p \rangle \tag{10a}$$
$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{10b}$$
$$\langle 1, \mu_0 \rangle = 1 \tag{10c}$$
$$\mu, \mu_p \in \mathcal{M}_+([0,T] \times X) \tag{10d}$$
$$\mu_0 \in \mathcal{M}_+(X_0). \tag{10e}$$

Both μ_0 and μ_p are probability measures by constraint (10c). The measures $\mu_0 = \delta_{x=x_0^*}$, $\mu_p = \delta_{t=t_p^*,\ x=x_p^*}$, and $\mu = \textbf{eval}([0, t_p^*], t \mapsto (t, x^*(t \mid x_0^*))$ are solutions to constraints (10b)-(10e). These measures yield an upper bound $p^* \geq P^*$, and there will be no relaxation gap ($p^* = P^*$) if the set $[0, T] \times X$ is compact (Sec. 2.3 of [5] and [9]). The moment-SOS hierarchy may be used to find a sequence of upper bounds to p^*. The method in [5] approaches the moment-SOS hierarchy from the dual side, involving SOS constraints in terms of an auxiliary function $v(t, x)$ (dual variable to constraint (10b)). The recovery procedure in [17] can be used to attempt extraction of near-optimal trajectories (x_0^*, t_p^*, x_p^*) if the moment matrices associated to μ_0 and μ_p are low-rank. Sublevel set methods presented in [5, 20] are a more robust method to extract near-optimal trajectories, but require a postprocessing optimization step after the moment-SOS LMIs have been solved.

2.3 Safety

This subsection reviews methods to verify that trajectories starting from $X_0 \subset X$ do not enter an unsafe set $X_u \subset X$. In Figure 2, the unsafe set $X_u = \{x \in \mathbb{R}^2 \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2,\ \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$ is the red half-circle to the bottom-left of trajectories.

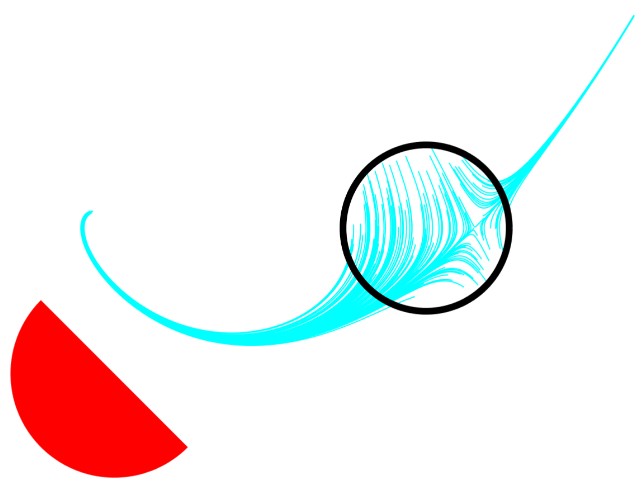


Figure 2: Trajectories of Flow system (5)

Sufficient conditions certifying safety can be obtained using barrier functions [14, 15]. However, these conditions do not provide a quantitative measurement for the safety of trajectories. Safety margins as introduced in [17] quantify the safety of trajectories through the use of maximin peak estimation. Assume that X_u is a basic semialgebraic set with description $X_u = \{x \mid p_i(x) \geq 0,\ i = 1, \ldots, N_u\}$. A point x is in X_u if all $p_i(x) \geq 0$. If at least one $p_i(x)$ remains negative for all points along trajectories $x(t \mid x_0)$, $x_0 \in X_0$, then no point starting from X_0 enters X_u and trajectories are therefore safe. The value $p^* = \max_i \min_{t,x_0} p_i(x(t \mid x_0))$ is called the safety margin, and a negative safety margin $p^* < 0$ certifies safety. The moment-SOS hierarchy (Appendix B) can be used to find upper bounds $p_d^* > p^*$ at degrees d, and safety is assured if any upper bound is negative $0 > p_d^* > p^*$. Figure 3 visualizes the safety margin for the Flow system (5), where the bound of $p^* \leq -0.2831$ was found at the degree-4 relaxation.

The safety margin of trajectories will generally change if the unsafe set X_u is reparameterized even in the same coordinate system. Let $q \leq 0$ and $s > 0$ be violation and scaling parameters for the enlarged unsafe set

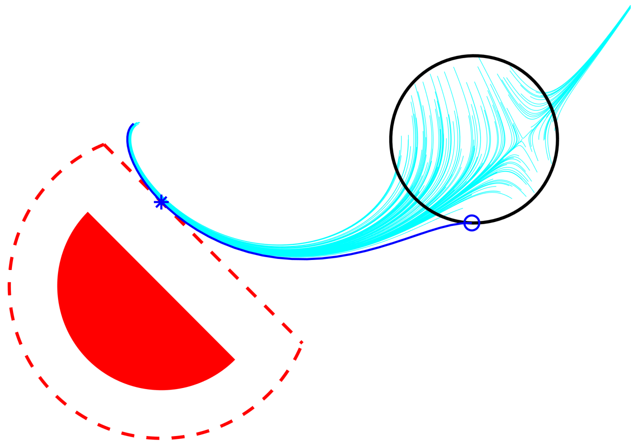


Figure 3: Flow system is safe, $p^* \leq -0.2831$

$(X_u^s)_q = \{x \mid q \leq 0.5^2 - x_1^2 + (x_2+0.7)^2,\ q \leq -s(x_1+x_2-0.7)\}$. The original unsafe set may be interpreted as $X_u = (X_u^{s=\sqrt{2}/2})_{q=0}$. Figure 4 visualizes contours of regions $(X_u^s)_q$ as q decreases from 0 down to -2 for sets with scaling parameters $s = 5$ and $s = 1$. The safety margins of trajectories with respect to X_u^s will vary as s changes, even as the same set X_u is represented in all cases. This is precisely the difficulty addressed in the present paper: developing scale invariant quantitative safety metrics.

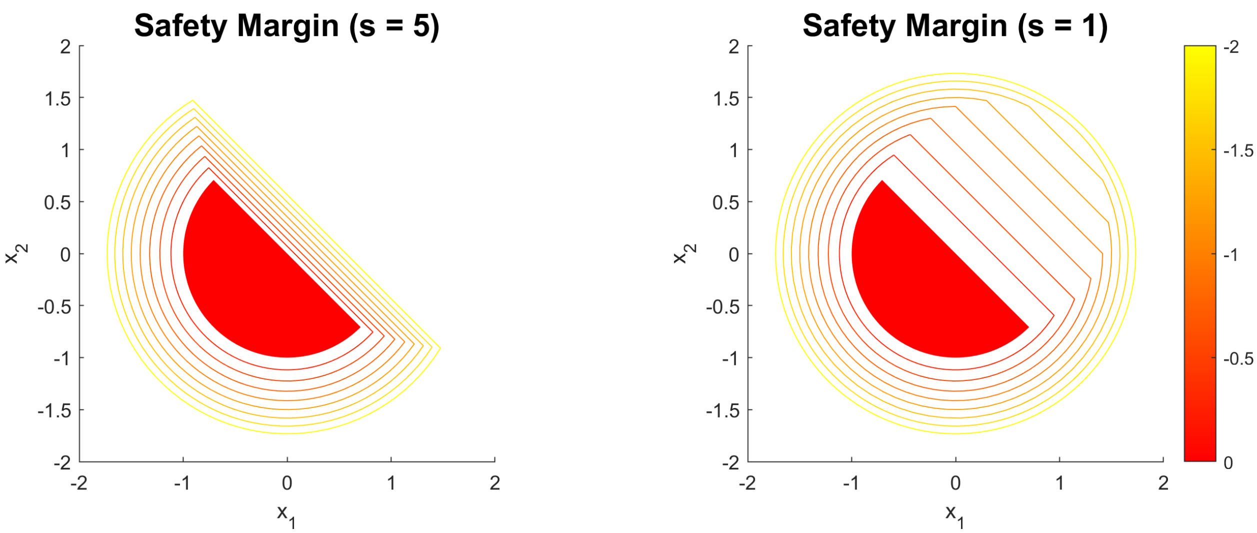


Figure 4: Safety margin scaling contours

3 Distance Estimation Program

The goal of this paper is to develop a computationally tractable framework to compute the worst case (over all initial conditions) distance of closest approach to an unsafe set. Specifically, we aim to solve the following problem.

Problem 1 (Distance Calculation)**.** *Given semi-algebraic initial condition* (X_o) *and unsafe* (X_u) *sets, solve the optimization problem* (1)*.*

In many practical situations, it is sufficient to obtain interpretable lower bounds on the minimum distance. Thus, the following problem is also of interest.

Problem 2 (Distance Estimation)**.** *Given semi-algebraic initial condition set (X_o), an unsafe (X_u) set, and a positive integer d (degree), find a lower bound $p^*_d \leq P^*$ to the solution of optimization* (1)*.*

As we will show in this paper (and under mild compactness and regularity conditions), a convergent sequence of lower bounds $\{p^*_d\}$ that rise to $\lim_{d\to\infty} p^*_d = P^*$ may be constructed where each bound p^*_d is obtained by solving a finite dimensional LMI.

An optimizing trajectory of the Distance program (1) may be described by a tuple $\mathcal{T}^* = (x^*_p,\ y^*\ x^*_0,\ t^*_p)$ as defined in Table 1.

Table 1: Characterization of optimal trajectory in distance estimation

x^*_p	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x^*_0	initial condition to produce x^*_p
t^*_p	time to reach x^*_p from x^*_0

The relationship between these quantities for an optimal trajectory of (1) is:

$$P^* = c(x^*_p; X_u) = c(x^*_p, y^*) = c(x(t^*_p \mid x^*_0), y^*). \tag{11}$$

Figure 5 plots the trajectory of closest approach to X_u in dark blue. This minimal L_2 distance is 0.2831, and the red curve is the level set of all points with a point-set distance 0.2831 to X_u. On the optimal trajectory, the blue circle is $x^*_0 \approx (1.489, -0.3998)$, the blue star is $x^*_p \approx (0, -0.2997)$, and the blue square is $y^* \approx (-0.2002, -0.4998)$. The closest approach of 0.2831 occurred at time $t^* \approx 0.6180$. Figure 6 plots the distance and safety margin contours for the set X_u.

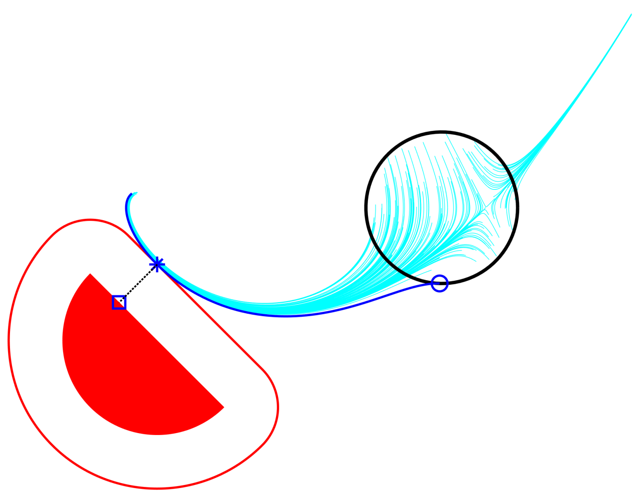


Figure 5: L_2 bound of 0.2831

3.1 Assumptions

The following assumptions are made in Program (1):

A1 The sets $[0, T],\ X,\ X_u$ are compact (Archimedean for numerical purposes).

A2 The function $f(t, x)$ is Lipschitz in each argument in the compact set $[0, T] \times X$.

A3 The cost $c(x, y)$ is C^0 in $X \times X_u$.

A4 If $x(t \mid x_0) \notin X$ for some $t \in [0, T],\ x_0 \in X_0$, then $x(t' \mid x_0) \notin X\ \forall t' \in [t, T]$.

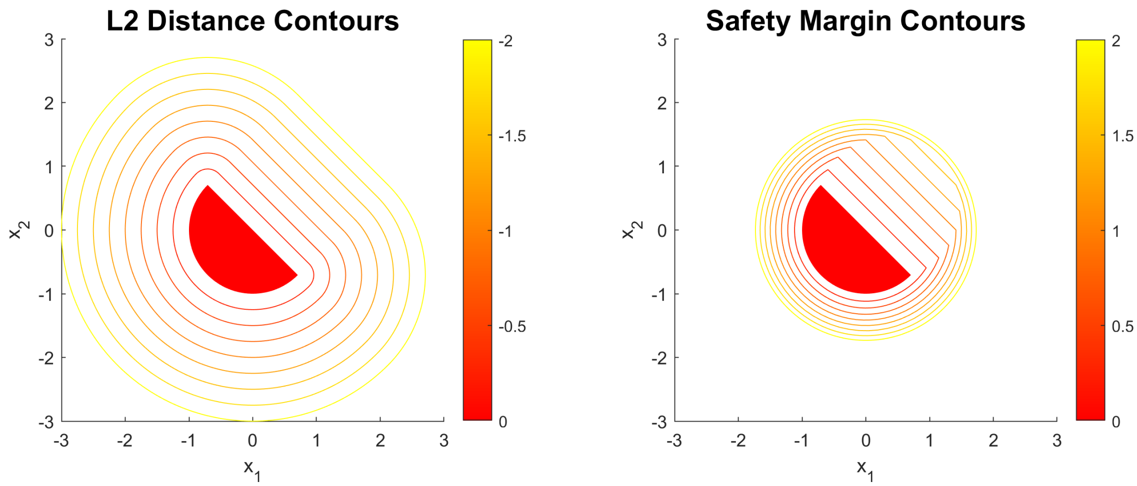


Figure 6: Comparison between L_2 distance and safety margin contours

Assumption A3 relaxes the requirement that c should be a metric, allowing for costs such as $\|x-y\|_2^2$ in addition to the metric $\|x-y\|_2$. The combination of A1 and A3 enforce that $c(x,y)$ is bounded inside $X \times X_u$ by the Weierstrass extreme value theorem. Assumption A4 requires that trajectories do not return to X after leaving, which is weaker than requiring that trajectories starting in X_0 remain in X for all $t \in [0,T]$.

3.2 Measure Program

The problem of $c^* = \min_{(x,y)\in X\times X_u} c(x,y)$ is identical to $c^* = \min_{(x,y)\in X\times X_u} \langle c(x,y), \delta_x \otimes \delta_y\rangle$ for Dirac measures $\delta_x \otimes \delta_y$. The Dirac restriction may be relaxed to minimization over the set of probability measures $c^* = \langle c(x,y), \eta\rangle, \eta \in \mathcal{M}_+(X \times X_u), \langle 1, \eta\rangle = 1$ with no change in the objective value c^*.

Theorem 3.1. *An infinite-dimensional LP in measures $(\mu_0, \mu_p, \mu, \eta)$ to bound from below the distance closest approach to X_u starting from X_0 (assuming that f is C^0 and A3-A4) is,*

$$
\begin{aligned}
p^* = \ \inf \quad & \langle c(x,y), \eta\rangle && \text{(12a)}\\
& \pi^x_\# \eta = \pi^x_\# \mu_p && \text{(12b)}\\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu && \text{(12c)}\\
& \langle 1, \mu_0\rangle = 1 && \text{(12d)}\\
& \eta \in \mathcal{M}_+(X \times X_u) && \text{(12e)}\\
& \mu_p, \ \mu \in \mathcal{M}_+([0,T]\times X) && \text{(12f)}\\
& \mu_0 \in \mathcal{M}_+(X_0). && \text{(12g)}
\end{aligned}
$$

Proof. Let $\mathcal{T} = (x_p, y, x_0, t_p) \in X \times X_u \times X_0 \times [0,T]$ be a tuple representing a trajectory with $x_p = x(t_p \mid x_0)$ achieving a distance $P = c(x_p, y)$. A set of measures (12e)-(12g) satisfying constraints (12b)-(12g) may be constructed from the tuple $\mathcal{T}$. The initial measure $\mu_0 = \delta_{x=x_0}$, the peak (free-time terminal) measure $\mu_p = \delta_{t=t_p} \otimes \delta_{x=x_p}$, and the joint measure $\eta = \delta_{x_p} \otimes \delta_{y=y}$ are all rank-one atomic probability measures. The occupation measure μ is the evaluation measure $\textbf{eval}([0,t_p], t \mapsto (t, x(t \mid x_0)))$ The distance objective (12a) for the tuple $\mathcal{T}$ may be evaluated as,

$$\langle c(x,y), \eta\rangle = \langle c(x,y), \delta_{x=x_p} \otimes \delta_{y=y}\rangle = c(x_p, y) = P. \tag{13}$$

Let $\{\mathcal{T}^k\}$ be a sequence of trajectory tuples with an monotonically decreasing sequence of achieved distances $\{P^k\}$. Each tuple $\mathcal{T}^k$ has a measure realization with objective P^k by the previously mentioned construction process. As $\lim_{k\to\infty} P^k = P^*$, the constraints (12b)-(12g) always remain feasible, which therefore proves the objective $p^* \leq P^*$.

□

Remark 1. *As a reminder, the term* $\pi^x_\#$ *from constraint* (12b) *is the operator performing* x*-marginalization. Constraint* (12b) *ensures that the* x*-marginals of* η *and* μ_p *are equal:* $\forall w \in C(X) : \langle w(x), \eta(x,y)\rangle = \langle w(x), \mu_p(t,x)\rangle$.

Lemma 3.2. *The masses of all measures in* (12) *are finite (bounded) if A1-A4 hold.*

Proof. Constraint (12d) imposes that $\langle 1, \mu_0\rangle = 1$, which further requires that $\langle 1, \mu_p\rangle = \langle 1, \mu_0\rangle = 1$ by constraint (12c) ($v(t,x) = 1$) and $\langle 1, \mu_p\rangle = \langle 1, \eta\rangle = 1$ ($w(x) = 1$). The occupation measure μ likewise has bounded mass with $\langle 1, \mu\rangle = \langle t, \mu^p\rangle < T$ by constraint (12c) ($v(t,x) = t$). □

Remark 2. *Problem* (12) *is convex. Letting* $\{\mathcal{T}^r\}_{r=1}^R$ *be a set of tuples with equal objectives* $P^r = P \ \forall r = 1..R$*, convex combinations of their constructed measures (with arbitrary weights* $w \in \mathbb{R}^R_+$ *such that* $\sum_r w_r = 1$*) will have identical objectives,*

$$\begin{aligned}\langle c, \eta\rangle &= \langle c, \textstyle\sum_r w_r \eta_r\rangle = \langle c, \textstyle\sum_r w_r \delta_{x^r} \otimes \delta_{y^r}\rangle \\ &= \textstyle\sum_r w_r c(x^r, y^r) = \left(\textstyle\sum_r w_r\right) P^* = P^*.\end{aligned} \tag{14}$$

3.3 Function Program

Dual variables $v(t,x) \in C^1([0,T]\times X)$, $w(x) \in C(X)$, $\gamma \in \mathbb{R}$ over constraints (12b)-(12d) must be introduced to derive the dual LP to (12). The Lagrangian $\mathscr{L}$ of problem (12) is:

$$\begin{aligned}\mathscr{L} = &\langle c(x,y), \eta\rangle + \langle v(t,x), \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu - \mu_p\rangle \\ &+ \langle w(x), \pi^x_\# \mu_p - \pi^x_\# \eta\rangle + \gamma(1 - \langle 1, \mu_0\rangle).\end{aligned} \tag{15}$$

Recalling that $\forall \eta \in \mathcal{M}_+(X \times Y)$, $f \in C(X)$ the relation that $\langle f(x), \eta(x,y)\rangle = \langle f(x), \pi^x_\# \eta(x)\rangle$ holds, the Lagrangian $\mathscr{L}$ in (15) may be reformulated as,

$$\begin{aligned}\mathscr{L} = &\gamma + \langle v(0,x) - \gamma, \mu_0\rangle + \langle c(x,y) - w(x), \eta\rangle \\ &+ \langle w(x) - v(t,x), \mu_p\rangle + \langle \mathcal{L}_f v(t,x), \mu\rangle.\end{aligned} \tag{16}$$

The dual of program (12) is provided by,

$$d^* = \sup_{\gamma, v, w} \ \inf_{\mu_0, \mu_p, \mu, \eta} \mathscr{L} \tag{17a}$$

$$= \sup_{\gamma \in \mathbb{R}} \quad \gamma \tag{17b}$$

$$v(0,x) \geq \gamma \qquad \forall x \in X_0 \tag{17c}$$

$$c(x,y) \geq w(x) \qquad \forall (x,y) \in X \times X_u \tag{17d}$$

$$w(x) \geq v(t,x) \qquad \forall (t,x) \in [0,T] \times X \tag{17e}$$

$$\mathcal{L}_f v(t,x) \geq 0 \qquad \forall (t,x) \in [0,T] \times X \tag{17f}$$

$$w \in C(X) \tag{17g}$$

$$v \in C^1([0,T] \times X). \tag{17h}$$

Theorem 3.3. *Strong duality is attained between problems* (12) *and* (17) *with attainment of optima under assumptions A1-A4.*

Proof. See Appendix A.

□

Remark 3. *The continuous function* $w(x)$ *is a lower bound on the point set distance* $c(x; X_u)$ *by constraint* (17d)*. The auxiliary function* $v(t,x)$ *is in turn a lower bound on* $w(x)$ *by constraint* (17e)*. This establishes a chain of lower bounds* $v(t,x) \leq w(x) \leq c(x; X_u)$ *holding* $\forall (t,x) \in [0,T] \times X$.

Theorem 3.4. *Under assumptions A1-A4,* $d^* = P^*$.

Proof. This proof will show that for every arbitrary $\delta > 0$, there a exists a feasible (γ, v, w) such that $P^* - \delta \leq d^* \leq P^*$. It therefore follows that $d^* = P^*$ under A1-A4.

The relation $d^* = p^*$ holds by strong duality from Theorem 3.3, and the bound $p^* \leq P^*$ is valid by measure construction in Theorem 3.1. Therefore, the bound from above $d^* \leq P^*$ is valid.

The function $c(x; X_u)$ is C^0, so an admissible choice of $w(x)$ is $w(x) = c(x; X_u)$. Appendix D of [5] provides a proof that (minimizing) peak estimation with C^0 state cost $w(x) = c(x; X_u)$ may be approximated to arbitrary accuracy by a C^1 auxiliary function $v(t, x)$. Such a v may be constructed by finding a function $W \in C^1([0, T] \times X)$ satisfying (a modification of equations D.2 and D.3 of [5] to account for inf rather than sup),

$$
\begin{aligned}
&\mathcal{L}_f W(t,x) \geq -\delta/(5T) && \forall (t,x) \in [0,T] \times X && \text{(18a)}\\
&w(x) \geq W(t,x) - (2/5)\delta && \forall (t,x) \in [0,T] \times X && \text{(18b)}\\
&W(0,x) \geq \gamma && \forall x \in X_0 && \text{(18c)}\\
&\gamma \geq P^* - (2/5)\delta, && && \text{(18d)}
\end{aligned}
$$

with v found from W by,

$$
v(t,x) = W(t,x) - (2/5)\delta - \delta/(5T)(T-t). \tag{19}
$$

The function $W(t, x)$ may be constructed from a flow map for trajectories of f by following the steps of Lemma D.2 of [5]. Feasible (γ, v, w) may be therefore found such that the bounds $P^* - \delta \leq d^* \leq P^*$ hold for every $\delta > 0$, which completes the proof. □

Corollary 1. *There exists a $\gamma \in \mathbb{R}$ and smooth functions $w \in C^\infty(X)$, $v \in C^\infty([0, T] \times X)$ such that $P^* - \delta \leq \gamma \leq P^*$ in* (17) *for every $\delta > 0$.*

Proof. For every $\epsilon > 0$, there exists a Stone-Weierstrass approximation $\bar{w}(x) \in \mathbb{R}[x]$ over the compact set X to the C^0 continuous function $c(x; X_u)$ such that $\sup_{x \in X} |c(x; X_u) - \bar{w}(x)| \leq \epsilon$. The function $w(x) = \bar{w}(x) - \epsilon$ is therefore a lower bound for $c(x; X_u)$ and satisfies constraints (17e) and (17g). A C^1 function $\tilde{v}(t, x)$ may be constructed from (18) and (19) (following process from Theorem 3.4) at a tolerance of $\delta' > 0$. This auxiliary function $\tilde{v}$ which may in turn be Stone-Weierstrass-approximated by $G \in \mathbb{R}[t, x]$ such that $\sup_{(t,x) \in [0,T] \times X} |G(t, x) - \tilde{v}(t, x)| \leq \epsilon$. The total approximation loss is $\delta = \delta' + 2\epsilon$, and letting $\delta', \epsilon \to 0$ leads to arbitrarily close approximations to P^* by smooth (v, w).

□

4 Finite-Dimensional Programs

This section presents finite-dimensional LMI and SDP truncations to the inifinite-dimensional LPs (12) and (17).

4.1 Approximation Preliminaries

We introduce notation and concepts about moments and SOS polynomials that will be used in subsequent finite-dimensional programs. Refer to Appendix B for further detail (e.g. Archimedean structure, moment-SOS hierarchy, conditions of convergence). A basic semialgebraic set $\mathbb{K} = \{x \mid g_i(x) \geq 0, \ i = 1, \ldots, N_c\}$ is a set formed by a finite set of bounded-degree polynomial constraints. The α-moment of a measure μ is $\mathbf{m}_\alpha = \langle x^\alpha, \mu \rangle$. The matrix $\mathbb{M}_d(\mathbb{K}\mathbf{m})$ formed by a moment sequence $\mathbf{m}$ is the block-diagonal matrix formed by $\text{diag}([\mathbf{m}_{\alpha+\theta}]_{\alpha,\theta \in \mathbb{N}^n_{\leq d}}, \forall k : [\sum_{\sigma \in \mathbb{N}^d} g_{k\sigma} \mathbf{m}_{\alpha+\theta+\sigma}]_{\alpha,\theta \in \mathbb{N}^n_{\leq d - \deg g_k/2}})$.

A polynomial $p(x)$ is SOS ($p(x) \in \Sigma[x]$) if there exists a finite integer s, a polynomial vector $v(x) \in \mathbb{R}[x]^s$, and a PSD matrix $Q \in \mathbb{S}^s_+$ such that $p(x) = v(x)^T Q v(x)$. SOS polynomials are nonnegative over $\mathbb{R}^n$. A polynomial is Weighted Sum of Squares (WSOS) over a set $\mathbb{K}$ (expressed as $p(x) \in \Sigma[\mathbb{K}]$ if there exists $\forall k = 0..N_c : \sigma_k \in \Sigma[x]$ such that $p(x) = \sigma_0(x) + \sum_k g_k(x) \sigma_k(x)$.

4.2 LMI Approximation

In the case where $c(x, y)$ and $f(t, x)$ are polynomial, (12) may be approximated with a converging hierarchy of SDPs. Assume that that X_0, X, and X_u are Archimedean basic semialgebraic sets, each defined by a finite number of bounded-degree polynomial inequality constraints $X_0 = \{g^0_k(x) \geq 0\}_{k=1}^{N_0}$, $X = \{g^X_k(x) \geq 0\}_{k=1}^{N_X}$, and $X_u = \{g^U_k(x) \geq 0\}_{k=1}^{N_U}$.

The polynomial inequality constraints for X_0, X, X_u are of degrees d^0_k, d_k, d^U_k respectively. The Liouville equation in (12c) enforces a countably infinite set of linear constraints indexed by all possible $\alpha \in \mathbb{N}^n,\ \beta \in \mathbb{N}$,

$$\langle x^\alpha, \mu_0\rangle\delta_{\beta 0} + \langle \mathcal{L}_f(x^\alpha t^\beta), \mu\rangle - \langle x^\alpha t^\beta, \mu_p\rangle = 0. \tag{20}$$

The expression $\delta_{\beta 0}$ is the Kronecker Delta taking a value $\delta_{\beta 0} = 1$ when $\beta = 0$ and $\delta_{\beta 0} = 0$ when $\beta \neq 0$. Let $(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}, \mathbf{m}^\eta)$ be moment sequences for the measures $(\mu_0, \mu_p, \mu, \eta)$. Define $\text{Liou}_{\alpha\beta}(\mathbf{m}^0, \mathbf{m}, \mathbf{m}^p)$ as the linear relation induced by (20) at the test function $x^\alpha t^\beta$ in terms of moment sequences. The polynomial metric $c(x, y)$ may be expressed as $\sum_{\alpha,\gamma} c_{\alpha\gamma} x^\alpha y^\gamma$ for multi-indices $\alpha, \gamma \in \mathbb{N}^n$. The complexity of dynamics f induces a degree $\tilde{d}$ as $\tilde{d} = d + \lceil \deg(f)/2\rceil - 1$. The degree-$d$ LMI relaxation of (12) with moment sequence variables $(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}, \mathbf{m}^\eta)$ is

$$
\begin{aligned}
p^*_d = \min \quad & \textstyle\sum_{\alpha,\gamma} c_{\alpha\gamma} \mathbf{m}^\eta_{\alpha\gamma}. && \text{(21a)}\\
& \mathbf{m}^\eta_{\alpha 0} = \mathbf{m}^p_{\alpha 0} \qquad \forall \alpha \in \mathbb{N}^n_{\leq 2d} && \text{(21b)}\\
& \text{Liou}_{\alpha\beta}(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}) = 0 \qquad \forall (\alpha, \beta) \in \mathbb{N}^{n+1}_{\leq 2d} && \text{(21c)}\\
& \mathbf{m}^0_0 = 1 && \text{(21d)}\\
& \mathbb{M}_d(X_0 \mathbf{m}^0) \succeq 0 && \text{(21e)}\\
& \mathbb{M}_d(([0, T] \times X)\mathbf{m}^p) \succeq 0 && \text{(21f)}\\
& \mathbb{M}_{\tilde{d}}(([0, T] \times X)\mathbf{m}) \succeq 0 && \text{(21g)}\\
& \mathbb{M}_d((X \times X_u)\mathbf{m}^\eta) \succeq 0. && \text{(21h)}
\end{aligned}
$$

Constraints (21b)-(21d) are finite-dimensional versions of constraints (12b)-(12g) from the measure LP.

Lemma 4.1. *The measures$(\mu_0, \mu_p, \mu, \eta)$ all have finite moments under Assumptions A1-A5.*

Proof. A sufficient condition for a measure $\tau \in \mathcal{M}_+(X)$ with compact support to be bounded is to have finite mass $\langle 1, \tau\rangle$. In our case, the support of all measures $(\mu_0, \mu_p, \mu, \eta)$ are compact sets if the region $[0, T] \times X \times X_u$ is Archimedean. Further, under Assumptions A1-A5, all of these measures have bounded mass (Lemma 3.2). This sufficiency is satisfied by all measures $(\mu_0, \mu_p, \mu, \eta)$. □

Theorem 4.2. *When T is finite and X_0, X, X_u are all Archimedean, the sequence of lower bounds $p^*_d \leq p^*_{d+1} \leq p^*_{d+2} \ldots$ will approach p^* as d tends towards ∞.*

Proof. This convergence is assured by Corollary 8 of [21], under the Archimedean assumption and Lemma 3.2.

□

Remark 4. *Non-differentiable but C^0 cost functions $c(x, y)$ may be approximated by polynomials $\tilde{c}(x, y)$ through the Stone-Weierstrass theorem in the compact set $X \times Y$. For every $\epsilon > 0$, there exists a $\tilde{c}(x, y) \in \mathbb{R}[x, y]$ such that $\max_{x\in X, y\in X_u} |c(x, y) - \tilde{c}(x, y)| \leq \epsilon$. Solving the peak estimation problem (12) with cost $\tilde{c}(x, y)$ as $\epsilon \to 0$ will yield convergent bounds to P^* with cost $c(x, y)$. Section 8.2 offers an alternative peak estimation problem for non-differentiable costs through the use of polyhedral lifts.*

4.3 Numerical Considerations

A moment matrix with n variables in degree d has dimension $\binom{n+d}{d}$. The sizes of moment matrices associated with a d relaxation of Problem (21) with state $x \in \mathbb{R}^n$, dynamics $f(t, x)$, and induced dynamic degree $\tilde{d}$ are listed in Table 2.

The computational complexity of solving the LMI (21) scales polynomially as the largest matrix size in Table 2, usually $\mathbb{M}_d(\mathbf{m}^\eta)$, except in cases where $f(t, x)$ has a high polynomial degree.

Table 2: Sizes of moment matrices in LMI (21)

Moment	$\mathbb{M}_d(\mathbf{m}^0)$	$\mathbb{M}_d(\mathbf{m}^p)$	$\mathbb{M}_{\tilde{d}}(\mathbf{m})$	$\mathbb{M}_d(\mathbf{m}^\eta)$
Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

Remark 5. *The measures μ_p and η may in principle be combined into a larger measure $\tilde{\eta} \in \mathcal{M}_+([0,T] \times X \times X_u)$. The Liouville equation* (12c) *would then read $\pi^{tx}_\# \tilde{\eta} = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu$, and a valid selection of $\tilde{\eta}$ given an optimal trajectory is $\tilde{\eta} = \delta_{t=t_p^*} \otimes \delta_{x=x_p^*} \otimes \delta_{y=y^*}$. The measure $\tilde{\eta}$ is defined over $2n+1$ variables, and the size of its moment matrix at a degree d relaxation is $\binom{1+2n+d}{d}$, as compared to $\binom{2n+d}{d}$ for η. We elected to split up the measures as μ_p and η to reduce the number of variables in the largest measure, and to ensure that the objective* (12a) *is interpretable as an earth-mover distance (from optimal transport literature[6]) between $\pi^x_\# \mu_p$ and a probability distribution over X_u (absorbed into $\pi^x_\# \eta$).*

Remark 6. *The distance problem* (1) *may also be treated as a peak estimation problem* (4) *with cost $p(x,y) = -c(x,y)$, initial set $X_0 \times X_u$, x-dynamics $\dot{x}(t) = f(t, x(t)$, and y-dynamics $\dot{y}(t) = 0$. The moment matrix $\mathbb{M}_d[\mathbf{m}]$ associated with this peak estimation problem's occupation measure (LMI relaxation of program* (10)*) would have size $\binom{1+2n+\tilde{d}}{d}$. This size is greater than any of the sizes written in Table 2.*

Remark 7. *The atom-extraction based recovery Algorithm 1 from [17] may be used to approximate near-optimal trajectories if the moment matrices $\mathbb{M}_d(\mathbf{m}^0)$, $\mathbb{M}_d(\mathbf{m}^p)$, and $\mathbb{M}_d(\mathbf{m}^\eta)$ are each low rank. If these matrices are all rank-one, then the near-optimal points (x_p, y, x_0, t_p) may be read directly from the moment sequences $(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}^\eta)$. The near optimal points from Figure 1 were recovered at the degree-4 relaxation of LMI* (21)*. The top corner of the moment matrices $\mathbb{M}_d(\mathbf{m}^0)$, $\mathbb{M}_d(\mathbf{m}^p)$, and $\mathbb{M}_d(\mathbf{m}^\eta)$ (containing moments of orders 0-2) have second-largest eigenvalues of* 1.87×10^{-5}, 8.82×10^{-6}, 5.87×10^{-7} *respectively, as compared to the largest eigenvalues of* 3.377, 1.472, 1.380.

4.4 SOS Approximation

The degree-d WSOS truncation of program (17) is,

$$
\begin{aligned}
d_d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma && \text{(22a)}\\
& v(0,x) - \gamma \in \Sigma[X_0]_{\leq 2d} && \text{(22b)}\\
& c(x,y) - w(x) \in \Sigma[X \times X_u]_{\leq 2d} && \text{(22c)}\\
& w(x) - v(t,x) \in \Sigma[[0,T] \times X]_{\leq 2d} && \text{(22d)}\\
& \mathcal{L}_f v(t,x) \in \Sigma[[0,T] \times X]_{\leq 2\tilde{d}} && \text{(22e)}\\
& w \in \mathbb{R}[x]_{\leq 2d} && \text{(22f)}\\
& v \in \mathbb{R}[t,x]_{\leq 2d}. && \text{(22g)}
\end{aligned}
$$

Theorem 4.3. *Strong duality holds with $p_k^* = d_k^*$ for all $k \in \mathbb{N}$ between* (21) *and* (22) *under assumptions A1-A5.*

Proof. Refer to Corollary 8 of [21] (Archimedean condition and bounded masses), as well as to the proof of Theorem 4 and Lemma 4 in Appendix D of [11]. □

5 Exploiting Correlative Sparsity

Many costs $c(x,y)$ exhibit an additively separable structure, such that c can be decomposed into the sum of new terms $c(x,y) = \sum_i c_i(x_i, y_i)$. Each term c_i in the sum is a function purely of (x_i, y_i). Examples include the L_p family of distance functions, such as the squared L_2 cost $c(x,y) = \sum_i (x_i - y_i)^2$. The theory of Correlative Sparsity in polynomial optimization, briefly reviewed below, can be used to substantially reduce the computational complexity entailed in solving the distance estimation LMIs when c is additively separable [22]. This decomposition does not require prior structure on the set $X \times X_u$. Other types of reducible structure (if applicable) include Term Sparsity [23], symmetry [24], and network dynamics [25]. These forms of structure may be combined if present, such as in Correlative and Term Sparsity [26].

5.1 Correlative Sparsity Background

Let $\mathbb{K} = \{x \mid g_k(x) \geq 0,\ k = 1, \ldots, N\}$ be an Archimedean basic semialgebraic set and $\phi(x)$ be a polynomial. The Correlative Sparsity Pattern (CSP) associated to $(\phi(x), g)$ is a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Each of the n vertices in $\mathcal{V}$ corresponds to a variable $x_1, \ldots, x_n$. An edge $(x_i, x_j) \in \mathcal{E}$ appears if variables x_i and x_j are multiplied together in a monomial in $\phi(x)$, or if they appear together in at least one constraint $g_k(x)$ [22].

The correlative sparsity pattern of $(\phi(x), g)$ may be characterized by sets I of variables and sets J of constraints. The p sets I should satisfy the following two properties:

1. (Coverage) $\bigcup_{j=1}^p I_j = \mathcal{V}$
2. (Running Intersection Property) For all $k = 1, \ldots, p-1$: $I_{k+1} \cap \bigcup_{j=1}^k I_j \subseteq I_s$ for some $s \leq k$

Equivalently, the sets I are the maximal cliques of a chordal extension of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ [27]. The sets $J = \{J_i\}_{i=1}^p$ are a partition over constraints $g_k(x) \geq 0$. The number k is in J_i for $k = 1, \ldots N_X$ if all variables involved the constraint polynomial $g_k(x)$ are contained within the set I_i. Let the notation $x(I_i)$ denote the variables in x that are members of the set I_i. A sufficient sparse representation of positivity certificates may be developed for $(\phi(x), g)$ satisfying an admissible correlative sparsity pattern (I, J) [28]:

$$\begin{aligned} &\phi(x) = \textstyle\sum_{i=1}^p \sigma_{i0}(x(I_i)) + \sum_{k \in J_i} \sigma_k(x(I_i)) g_k(x) \\ &\sigma_{i0}(x) \in \Sigma[x(I_i)] \qquad \sigma_k(x) \in \Sigma[x(I_i)] \qquad \forall i = 1, \ldots, p. \end{aligned} \tag{23}$$

Equation (23) is a sparse version of the Putinar certificate in (54). The sparse psatz (23) is additionally necessary if (I, J) satisfy the Running Intersection Property and a sparse Archimedean property holds: that there exist finite constants $R_i > 0$ for $i = 1..n$ such that $R_i^2 - \|x(I_i)\|_2^2$ is in the quadratic module (52) of constraints $Q[\{g_k\}_{k \in J_i}]$ [28].

5.2 Correlative Sparsity for Distance Estimation

Constraint (17d) will exhibit correlative sparsity when $c(x, y)$ is additively separable,

$$\textstyle\sum_{i=1}^n c_i(x_i, y_i) - w(x) \geq 0 \qquad\qquad \forall (x, y) \in X \times X_u. \tag{24}$$

The product-structure support set of Equation (24) may be expressed as,

$$\begin{aligned} X \times X_u = \{(x, y) \mid &g_1(x) \geq 0, \ldots g_{N_X}(x) \geq 0, \\ &g_{N_X+1}(y) \geq 0, \ldots g_{N_X+N_U}(y) \geq 0\}. \end{aligned} \tag{25}$$

The correlative sparsity graph of (24) is the graph Cartesian product of the complete graph K_n by K_2, and is visualized at $n = 4$ in Figure 7. Black lines imply that there is a link between variables. The black lines are drawn between each pair (x_i, y_i) from the cost terms c_i. The polynomial $w(x)$ involves mixed monomials of all variables $(x) = (x_1, x_2, x_3, x_4)$. Prior knowledge on the constraints of X_u are not assumed in advance, so the variables are $(y) = (y_1, y_2, y_3, y_4)$ joined together. A choice of CSP (I, J) associated with this system is,

$$\begin{aligned} I_1 &= \{x_1, x_2, x_3, x_4, y_1\} & J_1 &= \{1, \ldots, N_X\} \\ I_2 &= \{x_2, x_3, x_4, y_1, y_2\} & J_2 &= \varnothing \\ I_3 &= \{x_3, x_4, y_1, y_2, y_3\} & J_3 &= \varnothing \\ I_4 &= \{x_4, y_1, y_2, y_3, y_4\} & J_4 &= \{N_X + 1, \ldots, N_X + N_U\}. \end{aligned}$$

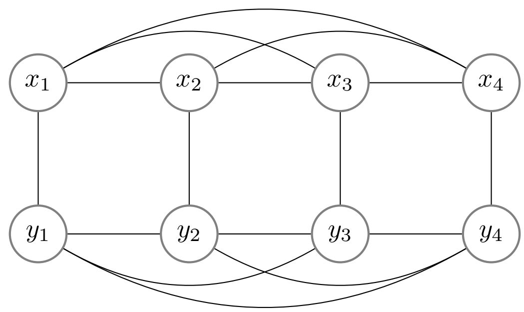


Figure 7: CSP with 4 states

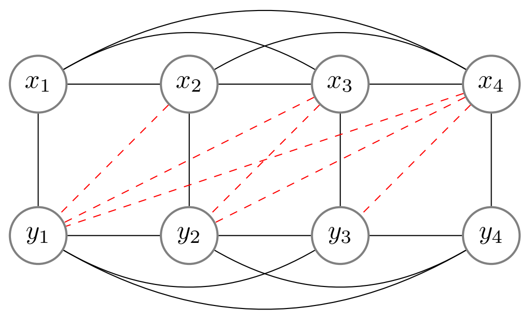


Figure 8: Chordal Extension of the CSP with 4-States

Figure 8 illustrates a chordal extension of the CSP graph with new edges displayed as red dashed lines. These new edges appear by connecting all variables in I_1 together in a clique, and by following a similar process for $I_2, \ldots I_4$.

For a unsafe distance bounding problem with a additively separable $c(x, y) = \sum_i c(x_i, y_i)$ with n states, the correlative sparsity pattern (I, J) is,

$$\begin{aligned} I_1 &= \{x_1, \ldots, x_n, y_1\} & J_1 &= \{1, \ldots, N_X\} \\ I_i &= \{x_i, \ldots x_n, y_1, \ldots y_i\} & J_i &= \varnothing, \qquad \forall i = 2, \ldots, n-1 \\ I_n &= \{x_n, y_1, \ldots, y_n\} & J_n &= \{N_X + 1, \ldots, N_X + N_U\}. \end{aligned} \tag{26}$$

A total of $(n-1)n/2$ new edges are added in the chordal extension. Letting $y_{1:i}$ be the collection of variables $(y_1, y_2, \ldots, y_i)$ for an index $i \in 1..n$ (and with a similar definition for $x_{i:n}$), a correlatively sparse certificate of positivity for constraint (17d) is,

$$\begin{aligned} \sum_{i=1}^{n} c_i(x_i, y_i) - w(x) = \sum_{i=1}^{n} \sigma_{i0}(x_{i:n}, y_{1:i}) + \sum_{k=1}^{N_X} \sigma_k(x, y_1) g_k(x) \\ + \sum_{k=N_X+1}^{N_X+N_U} \sigma_k(x_n, y) g_k(y), \end{aligned} \tag{27}$$

with sum-of-squares multipliers,

$$\begin{aligned} &\sigma_{i0}(x, y) \in \Sigma[x_{i:n}, y_{1:i}] & &\forall i = 1, \ldots, p \\ &\sigma_k(x, y) \in \Sigma[x, y_1] & &\forall k = 1, \ldots, N_X \\ &\sigma_k(x, y) \in \Sigma[x_n, y] & &\forall k = N_X + 1, \ldots, N_X + N_U. \end{aligned} \tag{28}$$

The application of correlative sparsity to the distance problem replaces constraint (22c) by (27).

Remark 8. *The CSP decomposition in* (26) *is nonunique. As an example, the following decompositions are all valid for* $n = 3$ *(satisfy Running Intersection Property),*

$$I_1 = \{x_1, x_2, x_3, y_1\} \qquad I'_1 = \{x_1, x_2, x_3, y_3\}$$
$$I_2 = \{x_2, x_3, y_1, y_3\} \qquad I'_2 = \{x_1, x_2, y_2, y_3\}$$
$$I_3 = \{x_2, y_1, y_2, y_3\} \qquad I'_3 = \{x_1, y_1, y_2, y_3\}$$

The original constraint (17d) is dual to the joint measure $\eta \in \mathcal{M}_+(X \times Y)$. Correlative sparsity may be applied to the measure program by splitting η into new measures $\eta_1 \in \mathcal{M}_+(X \times \mathbb{R}), \eta_n \in \mathcal{M}_+(\mathbb{R} \times X_u)$ and $\eta_i \in \mathcal{M}_+(\mathbb{R}^{n+1})$ for $i = 2, \ldots, n-1$ following the procedure in [28]. These measures will align on overlaps with $\pi_{\#}^{I_i \cap I_{i+1}} \eta_i = \pi_{\#}^{I_i \cap I_{i+1}} \eta_{i+1}, \ \forall i = 1, \ldots, n-1$. At a degree d relaxation, the moment matrix of η in (21) has size $\binom{2n+d}{d}$. Each of the n moment matrices of $\{\eta_i\}_{i=1}^n$ has a size of $\binom{n+1+d}{d}$. For example, a problem with $n = 4, d = 4$ will have a moment matrix for η of size $\binom{12}{4} = 495$ while the moment matrices for each of the $\eta_{(1:4)}$ are of size $\binom{9}{4} = 126$.

6 Shape Safety

The distance estimation problem may be extended to sets or shapes travelling along trajectories, bounding the minimum distance between points on the shape and the unsafe set. An example application is in quantifying safety of rigid body dynamics, for example finding the closest distance between all points on an airplane and points on a mountain.

6.1 Shape Safety Background

Let $X \subset \mathbb{R}^n$ be a region of space with unsafe set X_u, and $c(x, y)$ be a distance function. The state $\omega \in \Omega$ (such as position and angular orientation) follows dynamics $\dot{\omega}(t) = f(t, \omega)$ between times $t \in [0, T]$. A trajectory is $\omega(t \mid \omega_0)$ for some initial state $\omega_0 \in \Omega_0$. The shape of the object is a set S. There exists a mapping $A(s; \omega) : S \times \Omega \to X$ that provides the transformation between local coordinates on the shape (s) to global coordinates in X.

Examples of a shape traveling along trajectories are detailed in Figure 9. The shape $S = [-0.1, 0.1]^2$ is the pink square. The left hand plot is a pure translation after a $5\pi/12$ radian rotation, and the right plot involves a rigid body transformation.

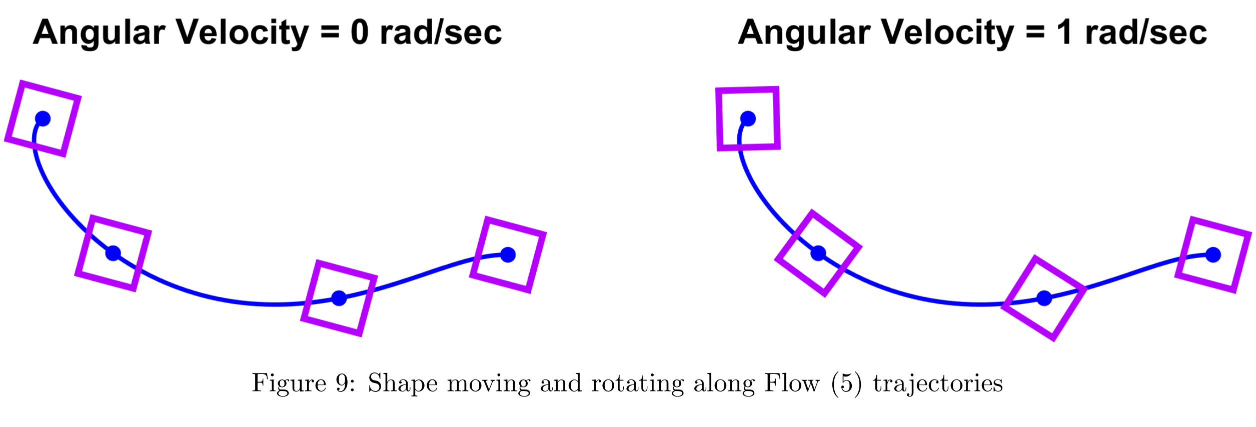


Figure 9: Shape moving and rotating along Flow (5) trajectories

The distance estimation task with shapes is to bound,

$$\begin{aligned} P^* = &\inf_{t,\, \omega_0 \in \Omega_0,\, s \in S,\, y \in X_u} c(A(s;\, \omega(t \mid \omega_0)),\ y) \\ &\dot{\omega}(t) = f(t, \omega), \quad \forall t \in [0, T]. \end{aligned} \tag{29}$$

For each trajectory in state $\omega(t \mid \omega_0)$, problem (29) ranges over all points in the shape $s \in S$ and points in the unsafe set $y \in X_u$ to find the closest approach. An optimal trajectory of the shape distance program may be expressed as $\mathcal{T}_s^* = (x_p^*, y^*, s^*, \omega_p^*, \omega_0^*, t_p^*)$ with,

$$P^* = c(x_p^*, y^*) = c(A(s^*; \omega_p^*), y^*) = c(A(s^*; \omega(t_p^* \mid \omega_0^*)), y^*).$$

6.2 Assumptions

The following assumptions are made in the Shape Distance program (29):

A1' The sets $[0, T],\ \Omega,\ \Omega_0,\ S,\ X,\ X_u$ are compact (Archimedean for numerical purposes).

A2' The function $f(t, \omega)$ is Lipschitz in each argument.

A3' The cost $c(x, y)$ is C^0.

A4' If $\omega(t \mid \omega_0) \notin \Omega$ for some $t \in [0, T],\ \omega_0 \in \Omega_0$, then $\omega(t \mid \omega_0) \notin \Omega\ \forall t' \in [t, T]$.

A5' If $\exists s \in S$ such that $A(s; \omega(t \mid \omega_0)) \notin X$ for some $t \in [0, T],\ \omega_0 \in \Omega_0$, then $A(s; \omega(t' \mid \omega_0)) \notin X\ \forall t' \in [t, T]$.

A6' The coordinate transformation function $A(s; \omega)$ is C^0.

6.3 Shape Distance Measure Program

Program (29) involves a distance objective $c(x, y)$, where the point $x = A(s; \omega)$ is given by a coordinate transformation between body coordinates s and the evolving orientation ω. In order to formulate a measure program to (29), a shape measure $\mu_s \in \mathcal{M}_+(S \times \Omega)$ may be added to bridge the gap between the changing orientation $\dot{\omega}$ and the comparison distance x. The shape measure contains information on the orientation ω and body coordinate s that yields the closest point x,

$$\langle z(\omega), \mu_p(t, \omega)\rangle = \langle z(\omega), \mu_s(s, \omega)\rangle \qquad \forall z \in C(\Omega) \tag{30a}$$
$$\langle w(x), \eta(x, y)\rangle = \langle w(A(s; \omega)), \mu_s(s, \omega)\rangle \qquad \forall w \in C(X). \tag{30b}$$

The shape measure μ_s chooses the worst-case body coordinate s and orientation ω from μ_p (30a), such that the point $x = A(s; \omega)$ comes as close as possible to the unsafe set's coordinate y (30b). We retain the coordinate x in order to decrease the computational complexity of the LMIs, as elaborated upon further in Remark 5.

The infinite dimensional measure program that lower bounds (29) is,

$$p^* = \ \inf \quad \langle c(x, y), \eta\rangle \tag{31a}$$
$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{31b}$$
$$\pi^\omega_\# \mu_p = \pi^\omega_\# \mu_s \tag{31c}$$
$$\pi^x_\# \eta = A(s; \omega)_\# \mu_s \tag{31d}$$
$$\langle 1, \mu_0\rangle = 1 \tag{31e}$$
$$\eta \in \mathcal{M}_+(X \times X_u) \tag{31f}$$
$$\mu_s \in \mathcal{M}_+(\Omega \times S) \tag{31g}$$
$$\mu_p,\ \mu \in \mathcal{M}_+([0, T] \times \Omega) \tag{31h}$$
$$\mu_0 \in \mathcal{M}_+(\Omega_0). \tag{31i}$$

Constraint (12b) in the original distance formulation is now split between (31c) and (31d) (which are equivalent to (30b) and (30a)).

Remark 9. *If the polynomial degree of the coordinate transformation $A(s; \omega)$ is κ, then the d-degree relaxation of problem (31) involves moments of μ_s up to order $2\kappa d$. For a system with N_ω orientation states and N_s shape variables, the size of the moment matrix for μ_s is then $\binom{N_s + N_\omega + \kappa d}{\kappa d}$. LMI constraints associated with μ_s can become bottlenecks to computation surpassing the contributions of μ and η as k increases.*

Remark 10. *Continuing the discussion Remark 5, the measures* $\mu_s(s,\omega)$ *and* $\eta(x,y)$ *may be combined together into a larger measure* $\eta_s(s,\omega,y) \in \mathcal{M}_+(S\times\Omega\times X_u)$ *with objective* $\inf\langle c(A(s;\omega),y),\eta_s\rangle$ *and constraint* $\pi^\omega_\# \mu_p = \pi^\omega_\# \eta_s$*. The moment matrix for* η_s *would have the generally intractable size* $\binom{N_s+N_\omega+n+\kappa d}{\kappa d}$*.*

Problem (31) inherits all convergence and duality properties of the original (12) under the appropriately modified set of assumptions A1'-A6'.

Theorem 6.1. *The Shape programs* (29) *and* (31) *are related by* $p^* \leq P^*$ *under the assumption that* f *is globally Lipschitz and A3'-A6'*

Proof. This proof will follow the same pattern as Theorem 3.1's proof. A set of measures that are feasible solutions for the constraints of (31) may be constructed for any trajectory $\mathcal{T}_s = (x_p, y, s, \omega_p, \omega_0, t_p)$. One choice of these measures are $\mu_0 = \delta_{\omega=\omega_0}$, $\mu_p = \delta_{t=t_p} \otimes \delta_{\omega=\omega_p}$, $\eta = \delta_{x=x_p} \otimes \delta_{y=y}$, $\mu_s = \delta_{s=s} \otimes \delta_{\omega=\omega_p}$ and μ as the evaluation measure $\textbf{eval}([0,t_p^*], t \mapsto (t, \omega(t \mid \omega_0^*)$ This construction is valid for any sequence of trajectories $\{T_s^k\}$ whose distances satisfy $\lim_{k\to\infty} P^k = P^*$, so the objectives are related by $p^* \leq P^*$. ☐

Lemma 6.2. *All measures in* (31) *have bounded mass under Assumption A1'.*

Proof. This follows from the steps of Lemma 3.2. The conditions hold that $1 = \langle 1, \mu_0\rangle = \langle 1, \mu_p\rangle$ (31b), $\langle 1, \mu_p\rangle = \langle 1, \mu_s\rangle$ (31c), $\langle 1, \mu_s\rangle = \langle 1, \eta\rangle$ (31d), and $\langle 1, \mu\rangle \leq T$ by (31b). ☐

6.4 Shape Distance Function Program

Defining a new dual function $z(\omega)$ against constraint (31c) (also observed in (30a)), the Lagrangian of problem (31) is:

$$\begin{aligned} \mathscr{L} = &\langle c(x,y), \eta\rangle + \langle v(t,x), \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu - \mu_p\rangle \\ &+ \langle z(\omega), \pi^\omega_\#(\mu_p - \mu_s)\rangle + \gamma(1 - \langle 1, u_0\rangle) \\ &+ \langle w(x), A(s;\omega)_\# \mu_s - \pi^x_\# \eta\rangle. \end{aligned} \tag{32}$$

The Lagrangian in (32) may be manipulated into,

$$\begin{aligned} \mathscr{L} = &\gamma + \langle c(x,y) - w(x), \eta\rangle + \langle v(0,\omega) - \gamma, \mu_0\rangle \\ &+ \langle \mathcal{L}_f v(t,\omega), \mu\rangle + \langle z(\omega) - v(t,\omega), \mu_p\rangle \\ &+ \langle w(A(s;\omega)) - z(\omega), \mu_s\rangle. \end{aligned} \tag{33}$$

The dual of program (31) provided by minimizing the Lagrangian (33) with respect to $(\eta, \mu_s, \mu_p, \mu, \mu_0)$ is,

$$d^* = \sup_{\gamma \in \mathbb{R}} \quad \gamma \tag{34a}$$

$$v(0,\omega) \geq \gamma \qquad \forall x \in \Omega_0 \tag{34b}$$

$$c(x,y) \geq w(x) \qquad \forall (x,y) \in X \times X_u \tag{34c}$$

$$w(A(s;\omega)) \geq z(\omega) \qquad \forall (s,\omega) \in S \times \Omega \tag{34d}$$

$$z(\omega) \geq v(t,\omega) \qquad \forall (t,\omega) \in [0,T] \times \Omega \tag{34e}$$

$$\mathcal{L}_f v(t,\omega) \geq 0 \qquad \forall (t,\omega) \in [0,T] \times \Omega \tag{34f}$$

$$w \in C(X), \ z \in C(\Omega) \tag{34g}$$

$$v \in C^1([0,T] \times X). \tag{34h}$$

Theorem 6.3. *Problems* (31) *and* (34) *are strongly dual under assumptions A1'-A6'.*

Proof. This holds by extending the proof of Theorem 3.3 found in Appendix A and applying Theorem 2.6 of [29]. ☐

Remark 11. *Program 34 imposes that a chain of lower bounds* $v(t,\omega) \leq z(\omega) \leq w(A(s;\omega)) \leq c(A(s;\omega)), y)$ *holds for all* $(s,\omega,t,y) \in S \times \Omega \times [0,T] \times X_u$ *(similar in principle to Remark 3).*

Theorem 6.4. *Under assumptions A1'-A6', there exists a feasible* (γ, v, w, z) *such that* $P^* + \delta \leq d^* \leq P^*$ *between* (34) *and* (29).

Proof. This proof follows from the steps of Theorem 3.4. The following C^0 functions are feasible for constraints (34c) and (34d),

$$w(x) = \inf_{y \in X_u} c(x, y), \qquad\qquad z(\omega) = \inf_{(s,y) \in S \times X_u} c(A(s; \omega), y).$$

The function $z(\omega)$ is C^0 since it is generated by the infimum of the composition of two C^0 functions c and A. The auxiliary function $v(t, \omega)$ may be chosen to solve the peak minimization problem $\min_{t,\omega_0} z(\omega(t \mid \omega_0))$ along trajectories starting from Ω_0 up to δ-optimality (by the Flow map construction of [5] as used in Theorem 3.4).

□

Corollary 2. *Problem* (29) *may be approximated up to* δ*-optimality by smooth functions* (v, w, z) *under Assumptions A1'-A6'.*

Proof. The steps of Corollary 1 may be traversed here. For any $\epsilon > 0$, Stone Weierstrass approximations $\bar{w} \in \mathbb{R}[x], \bar{z} \in \mathbb{R}[\omega]$ may be constructed such that $\sup_{x \in X} |\bar{w}(x) - \inf_{y \in X_u} c(x, y)| \leq \epsilon$ and $\sup_{\omega \in \Omega} |\bar{z}(\omega) - \inf_{(s,y) \in S \times X_u} c(A(s; \omega), y)| \leq \epsilon$. Feasible functions $w(x) = \bar{w}(x) - \epsilon$ and $z(\omega) = \bar{z}(\omega) - 2\epsilon$ may be chosen, with the polynomial auxiliary function $v(t, \omega) \in \mathbb{R}[t, \omega]$ arising from a δ'-approximation to the peak minimization problem with objective $z(\omega)$. The corollary is proven by setting $\delta = \delta' + 2\epsilon$. □

Remark 12. *We briefly note that the LMI formulation of* (31) *will converge to* P^* *under assumptions A1'-A6' if all sets* $[0, T], X, X_u, \Omega_0, \Omega, S$ *are Archimedean and if* $f(t, \omega) \in \mathbb{R}[t, \omega]$, $A(s; \omega) \in \mathbb{R}[s, \omega]$ *(from Theorem 4.2). Constraint* (30b) *induces a linear expression in moments for* (η, μ_s) *for each* $\alpha \in \mathbb{N}^n$: $\langle x^\alpha, \eta \rangle = \langle A(s; \omega)^\alpha, \mu_s \rangle$.

7 Numerical Examples

All code was written in Matlab 2021a, and is publicly available at the link https://github.com/Jarmill/distance. The LMIs were formulated by Gloptipoly3 [30] through a Yalmip interface [31], and were finally solved using Mosek [32]. The experimental platform was an Intel i9 CPU with a clock frequency of 2.30 GHz and 64.0 GB of RAM. The squared-L_2 cost $c(x, y) = \sum_i (x_i - y_i)^2$ is used in solving Problem (21) unless otherwise specified. The documented bounds are the square roots of the returned quantities, yielding lower bounds to the L_2 distance.

7.1 Flow System with Moon

The half-circle unsafe set in Figure 6 is a convex set. The moon-shaped unsafe set X_u in Figure 10 is the nonconvex region outside the circle with radius 1.16 centered at $(0.6596, 0.3989)$ and inside the circle with radius 0.8 centered at $(0.4, -0.4)$. The dotted red line demonstrates that trajectories of the Flow system would be deemed unsafe if X_u was relaxed to its convex hull.

The L_2 distance bound of 0.1592 in Figure 11 was found at the degree-5 relaxation of Problem (21). The moment matrices $\mathbb{M}_d(m^0)$, $\mathbb{M}_d(m^p), \mathbb{M}_d(m^\eta)$ at $d = 5$ were approximately rank-1 and near-optimal trajectories were successfully extracted. This near-optimal trajectory starts at $x_0^* \approx (1.489, -0.3998)$ and reaches a closest distance between $x_p^* \approx (1.113, -0.4956)$ and $y^* \approx (1.161, -0.6472)$ at time $t_p^* \approx 0.1727$. The distance bounds computed at the first five relaxations are $L_2^{1:5} = [1.487 \times 10^{-4}, 2.433 \times 10^{-4}, 0.1501, 0.1592, 0.1592]$.

7.2 Twist System

The Twist system is a three-dimensional dynamical system parameterized by matrices A and B,

$$\dot{x}_i(t) = \textstyle\sum_j A_{ij} x_j - B_{ij}(4x_j^3 - 3x_j)/2, \tag{35}$$

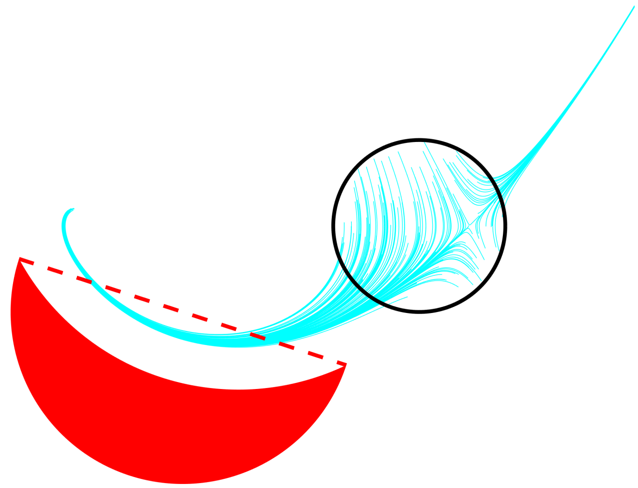


Figure 10: Collision if X_u is relaxed to its convex hull.

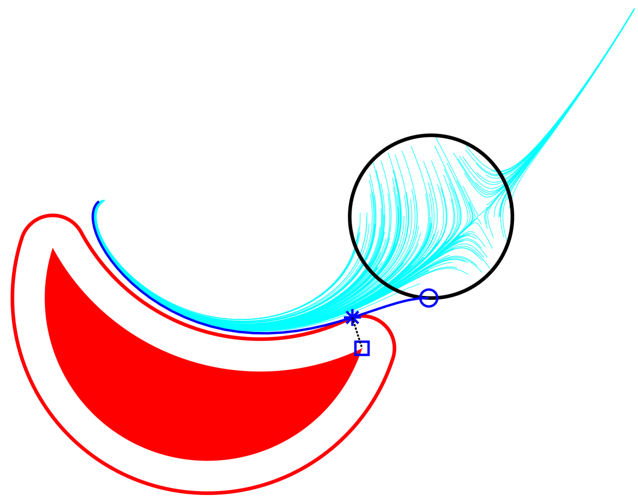


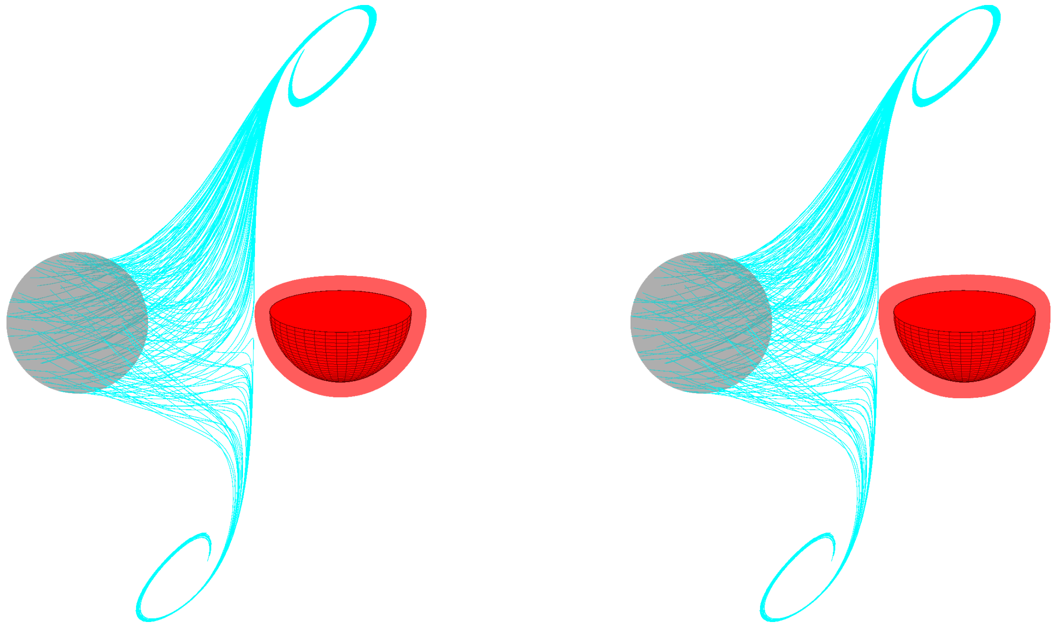
Figure 11: L_2 bound of 0.1592

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{36}$$

The cyan curves in each panel of Figure 12 are plots of trajectories of the Twist system between times $t \in [0, 5]$. These trajectories start at the $X_0 = \{x \mid (x_1+0.5)^2+x_2^2+x_3^2 \leq 0.2^2\}$ which is pictured by the grey spheres. The unsafe set $X_u = \{x \mid (x_1 - 0.25)^2 + x_2^2 + x_3^2 \leq 0.2^2,\ x_3 \leq 0\}$ is drawn in the red half-spheres.

The red shell in Figure 12a is the cloud of points within an L_2 distance of 0.0427 of X_u, as found through a degree 5 relaxation of (21). Figure 12b involves an L_4 contour of 0.0411, also found at a degree 5 relaxation. The first few distance bounds for the L_2 distance are $L_2^{1:5} = [0, 0, 0.0336, 0.0425, 0.0427]$, and for the L_4 distance are $L_4^{2:5} = [0, 0.0298, 0.0408, 0.0413]$. Fourth order moments are required for the L_4 metric, so the $L_4^{2:5}$ sequence starts at degree 2.

Table 3 and 4 lists the L_2 bounds and runtimes respectively generated by a distance estimation task between trajectories and the half sphere of the above L_2 Twist system example. The high-degree relaxations (orders 4 and 5) are significantly faster as found by the sparse LMI (dual to the sparse SOS with putinar expression (27)) as compared to the standard LMI in (21). The certifiable L_2 bounds returned are roughly equivalent between relaxations.



(a) L_2 bound of 0.0427 (b) L_4 bound of 0.0411

Figure 12: Distance contours at order-5 relaxation for the Twist system (35)

Table 3: L_2 bounds for the Twist Example

order	2	3	4	5	6
Standard LMI (21)	0.000	0.0313	0.0425	0.0429	0.0429
Sparse LMI with (27)	0.000	0.0311	0.0424	0.0430	0.0429

7.3 Shape Examples

Figure 13 visualizes a near-optimal trajectory of the shape distance estimation for orientations $\omega \in \mathbb{R}^2$ evolving as the flow system with an initial condition $\Omega_0 = \{\omega : (\omega_1 - 1.5)^2 + \omega_2^2 \leq 0.4^2\}$. Suboptimal trajectories were suppressed in visualization to highlight the shape structure and attributes of the near-optimal trajectory. The degree-1 coordinate transformation function A for pure translation with a constant rotation of $5\pi/12$ is,

$$A(s;\omega) = \begin{bmatrix} \cos(5\pi/12)s_1 - \sin(5\pi/12)s_2 + \omega_1 \\ \cos(5\pi/12)s_1 + \sin(5\pi/12)s_2 + \omega_2 \end{bmatrix}. \tag{37}$$

This near-optimal trajectory with an L_2 distance bound of 0.1465 was found at a degree-4 relaxation of Problem (31). The near-optimal trajectory is described by $\omega_0^* \approx (1.489, -0.3887)$, $t_p^* \approx 3.090$, $\omega_p^* \approx (-0.1225, -0.3704)$, $s^* \approx (-0.1, 0.1)$, $x_p^* \approx (0, -0.2997)$, and $y^* \approx (-0.2261, -0.4739)$. The first five distance bounds are $L_2^{1:5} = [1.205 \times 10^{-4}, 4.245 \times 10^{-4}, 0.1424, 0.1465, 0.1465]$.

In the following example, the shape S is now rotating at an angular velocity of 1 radian/second, as shown in the right panel of Fig. 9. The orientation $\omega \in SE(2)$ may be expressed as a semialgebraic lift through $\omega \in \mathbb{R}^4$ with trigonometric terms $\omega_3^2 + \omega_4^2 = 1$. The dynamics for this system are,

$$\dot{\omega} = \begin{bmatrix} \omega_2 \\ -\omega_1 - \omega_2 + \frac{1}{3}\omega_1^3 \\ -\omega_4 \\ \omega_3 \end{bmatrix}. \tag{38}$$

The degree-2 coordinate transformation associated with this orientation is,

$$A(s;\omega) = \begin{bmatrix} \omega_3 s_1 - \omega_4 s_2 + \omega_1 \\ \omega_3 s_1 + \omega_4 s_2 + \omega_2 \end{bmatrix}. \tag{39}$$

Table 4: Time in seconds for the Twist Example in Section 7.2

order	2	3	4	5	6
Standard LMI (21)	0.32	1.92	47.55	502.29	4028.94
Sparse LMI with (27)	0.31	1.19	7.07	45.89	184.42

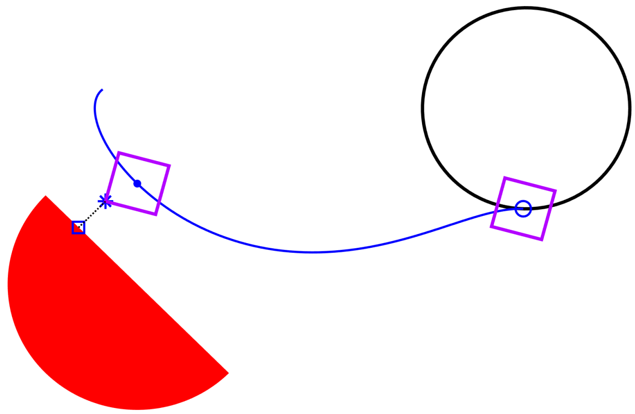


Figure 13: Translation, L_2 bound of 0.1465

The shape measure $\mu_s \in \mathcal{M}_+(S \times \Omega)$ is distributed over 6 variables. The size of μ_s's moment matrix with $k = 2$ at degrees 1-4 is $[28, 210, 924, 3003]$. The first three distance bounds are $L_2^{1:3} = [2.9158 \times 10^{-5}, 0.059162, 0.14255]$, and MATLAB runs out of memory on the experimental platform at degree 4. A successful recovery is achieved at the degree 3 relaxation, as pictured in Figure 14. This rotating-set near-optimal trajectory is encoded by $\omega_0^* \approx (1.575, -0.3928, 0.2588, 0.9659)$, $t_p^* \approx 3.371$, , $s^* \approx (-0.1, 0.1)$, $x_p^* \approx (-0.1096, -0.3998)$, $\omega_p^* \approx (-0.0064, -0.2921, -0.0322, -0.9995)$, and $y^* \approx (-0.2104, -0.4896)$. Computing this degree-3 relaxation required 75.43 minutes.

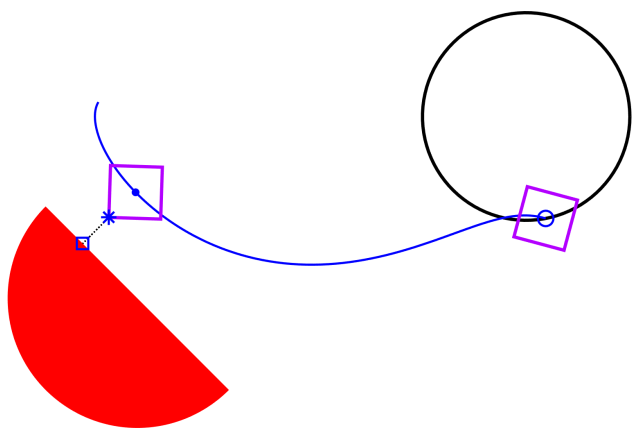


Figure 14: Rotation, L_2 bound of 0.1425

8 Extensions

This section presents modifications to the distance estimation programs in order to handle systems with uncertainties and distance functions c generated by polyhedral norms.

8.1 Uncertainty

Distance estimation can be extended to systems with uncertainty. For the sake of simplicity, this section is restricted to time-dependent uncertainty. Assume that $H \subset \mathbb{R}^{N_h}$ is a compact set of plausible values of uncertainty, and the uncertain process $h(t), \forall t \in [0, T]$ may change arbitrarily in time within H [33]. The distance estimation problem with time-dependent uncertain dynamics is,

$$\begin{aligned} P^* = & \inf_{t,\, x_0,\, y,\, h(t)} c(x(t \mid x_0, h(t)), y) \\ & \dot{x}(t) = f(t, x, h(t)) & & \forall t \in [0, T] \\ & h(t) \in H & & \forall t \in [0, T] \\ & x_0 \in X_0, \;\; y \in X_u. \end{aligned} \tag{40}$$

The process $h(t)$ acts as an adversarial optimal control aiming to steer $x(t)$ as close to X_u as possible. The occupation measure μ may be extended to a Young measure (relaxed control) $\mu \in \mathcal{M}_+([0, T] \times X \times H)$ [34, 10].

The Liouville equation (12c) may be replaced by $\mu_p = \delta_0 \otimes \mu_0 + \pi^{tx}_{\#}\mathcal{L}_f^{\dagger}\mu$, which should be understood to read $\langle v(t, x), \mu_p\rangle = \langle v(0, x), \mu_0\rangle + \langle \partial_t v(t, x) + f(t, x, h) \cdot \nabla_x v(t, x), \mu\rangle$ for all test functions $v \in C^1([0, T] \times X)$. Any trajectory with uncertainty process $h(t)$ may be represented by a tuple $(x_0, x_p, t_p, y, h(\cdot))$. This trajectory admits a measure representation similar to the proof of 3.1, where the occupation measure μ is $\mathbf{eval}([0, t_p], t \mapsto (t, x(t \mid x_0), h(t)))$. The work in [33] applies a collection of existing uncertainty structures to peak estimation problems (time-independent, time-dependent, switching-type, box-type), and all of these structures may be applied to distance estimation.

To illustrate these ideas, consider the following Flow system with time-dependent uncertainty:

$$\dot{x} = \begin{bmatrix} x_2 \\ (-1 + h)x_1 - x_2 + \frac{1}{3}x_1^3 \end{bmatrix} \qquad h \in [-0.25, 0.25]. \tag{41}$$

An L_2 distance bound of 0.1691 is computed at the degree 5 relaxation of the uncertain distance estimation program, as visualized in Figure 15. The first five distance bounds are $L_2^{1:5} = [5.125 \times 10^{-5}, 1.487 \times 10^{-4}, 0.1609, 0.1688, 0.1691]$.

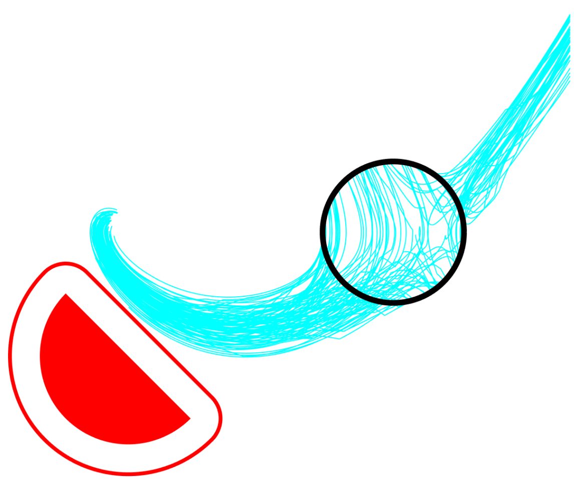


Figure 15: Uncertain Flow (41), L_2 bound of 0.1691

8.2 Polyhedral Norm Penalties

The infinite dimensional LP (12) is valid for all continuous costs $c(x, y) \in C(X^2)$, but its LMI relaxation can only handle polynomial costs $c(x, y) \in \mathbb{R}[x, y]$. The L_p distance is defined as $\|x - y\|_p = \sqrt[p]{\sum_i |x_i - y_i|^p}$ when p is finite and $\|x - y\|_\infty = \max_i |x_i - y_i|$ for p infinite. The cost $\|x - y\|_p^p$ is polynomial when p is finite and even; otherwise the L_p distance has a piecewise definition in terms of absolute values. The theory of convex (LP) lifts may be used to interpret piecewise constraints into valid LMIs [35, 36]. Slack variables $q \in \mathbb{R}$ (or $q_i \in \mathbb{R}$ as appropriate) may be added to form enriched infinite dimensional LPs. The objective $\langle c, \eta \rangle$ from (12a) could be replaced by the following terms for the examples of L_∞, L_1, and L_3 distances:

$$\|x - y\|_\infty \qquad \min \quad q \qquad -q \le \langle x_i - y_i, \eta \rangle \le q \qquad \forall i = 1, \dots, n \tag{42a}$$

$$\|x - y\|_1 \qquad \min \quad \textstyle\sum_i q_i \qquad -q_i \le \langle x_i - y_i, \eta \rangle \le q_i \qquad \forall i = 1, \dots, n, \tag{42b}$$

$$\|x - y\|_3^3 \qquad \min \quad \textstyle\sum_i q_i \qquad -q_i \le \langle (x_i - y_i)^3, \eta \rangle \le q_i \qquad \forall i = 1, \dots, n. \tag{42c}$$

Distances induced by polyhedral norms can be included through this lifting framework [37]. Figure 16 visualizes the near-optimal trajectory for a minimum L_1 distance bound of 0.4003 (cost (42c)) at degree 4. This trajectory starts at $x_0^* \approx (1.489, -0.3998)$ and reaches the closest approach between $x_p^* \approx (0, -0.2997)$ and $y^* \approx (-0.1777, -0.5223)$ at time $t^* \approx 0.6181$ units. The first five L_1 distance bounds are $L_1^{1:5} = [3.179 \times 10^{-9}, 4.389 \times 10^{-8}, 0.3146, 0.4003, 0.4003]$.

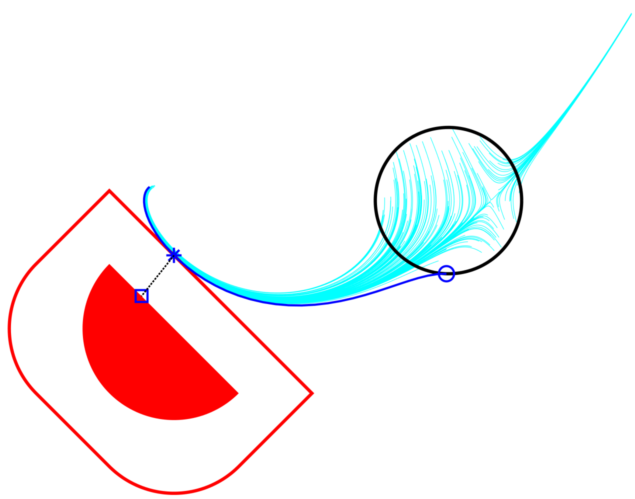


Figure 16: L_1 bound of 0.4003

9 Conclusion

This paper presented an infinite dimensional linear program in occupation measures to approximate the distance estimation problem. The LP objective is equal to the distance of closest approach between points along trajectories and points on the unsafe set under mild compactness and regularity conditions. Finite-dimensional truncations of this LP yield a converging sequence of SDP lower bounds to the minimal distance under further conditions (Archimedean). The distance estimation problem can be modified to accommodate dynamics with uncertainty, piecewise distance functions, and movement of shapes along trajectories. Future work includes formulating and implementing control policies to maximize the distance of closest approach to the unsafe set while still reaching a terminal set within a specified time.

A Proof of Strong Duality in Theorem 3.3

This proof will follow the method used in Theorem 2.6 of [29] to prove duality.

The two programs (12) and (17) will be posed as a pair of standard-form infinite dimensional LPs using notation from [29]. The following spaces may be defined:

$$\begin{aligned}\mathcal{X}' &= C(X_0) \times C([0,T] \times X)^2 \times C(X \times X_u) \\ \mathcal{X} &= \mathcal{M}(X_0) \times \mathcal{M}([0,T] \times X)^2 \times \mathcal{M}(X \times X_u).\end{aligned} \tag{43}$$

The nonnegative subcones of $\mathcal{X}'$ and $\mathcal{X}$ respectively are,

$$\begin{aligned}\mathcal{X}'_+ &= C_+(X_0) \times C_+([0,T] \times X)^2 \times C_+(X \times X_u) \\ \mathcal{X}_+ &= \mathcal{M}_+(X_0) \times \mathcal{M}_+([0,T] \times X)^2 \times \mathcal{M}_+(X \times X_u).\end{aligned} \tag{44}$$

The cones $\mathcal{X}'_+$ and $\mathcal{X}_+$ in (44) are topological duals under assumption A1, and the measures from (12e)-(12g) satisfy $\boldsymbol{\mu} = (\mu_0, \mu_p, \mu, \eta) \in \mathcal{X}_+$.

The spaces $\mathcal{Y}$ and $\mathcal{Y}'$ may be defined as,

$$\mathcal{Y} = C(X) \times C^1([0,T] \times X) \times \mathbb{R} \tag{45}$$

$$\mathcal{Y}' = \mathcal{M}(X) \times C^1([0,T] \times X)' \times 0. \tag{46}$$

We express $\mathcal{Y}_+ = \mathcal{Y}$ and $\mathcal{Y}'_+ = \mathcal{Y}'$ to maintain a convention with [29] given there are no affine-inequality constraints in (12).

The arguments $\boldsymbol{\ell} = (w, v, \gamma)$ from problem (17) are members of the set $\mathcal{Y}'_+$.

The linear operators $\mathcal{A}' : \mathcal{Y}'_+ \to \mathcal{X}'_+$ and $\mathcal{A} : \mathcal{X}_+ \to \mathcal{Y}_+$ induced from constraints (12b)-(12d) may be defined as,

$$\begin{aligned}\mathcal{A}(\boldsymbol{\mu}) &= [\pi^x_\# \mu_p - \pi^x_\# \eta, \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu - \mu_p, \langle 1, \mu_0 \rangle] \\ \mathcal{A}'(\boldsymbol{\ell}) &= [v(0,x) - \gamma, w(x) - v(t,x), \mathcal{L}_f v(t,x), \\ &\qquad c(x,y) - w(x)].\end{aligned} \tag{47}$$

The last pieces needed to convert (12) into a standard-form LP are the cost vector $\mathbf{c} = [0,0,0,c(x,y)]$ and the answer vector $\mathbf{b} = [0,0,1] \in \mathcal{Y}'$.

Problem (12) is therefore equivalent to (with $\langle \mathbf{c}, \boldsymbol{\mu} \rangle = \langle c, \eta \rangle$),

$$p^* = \inf_{\boldsymbol{\mu} \in \mathcal{X}_+} \langle \mathbf{c}, \boldsymbol{\mu} \rangle \qquad \mathbf{b} - \mathcal{A}(\boldsymbol{\mu}) \in \mathcal{Y}_+. \tag{48}$$

The dual LP to (48) in standard form is (with $\langle \boldsymbol{\ell}, \mathbf{b} \rangle = \gamma$),

$$d^* = \sup_{\boldsymbol{\ell} \in \mathcal{Y}'_+} \langle \boldsymbol{\ell}, \mathbf{b} \rangle \qquad \mathcal{A}'(\boldsymbol{\ell}) - \mathbf{c} \in \mathcal{X}_+. \tag{49}$$

The operators $\mathcal{A}$ and $\mathcal{A}'$ are adjoints with $\langle \mathcal{A}(\boldsymbol{\ell}), \boldsymbol{\mu} \rangle = \langle \boldsymbol{\ell}, \mathcal{A}'(\boldsymbol{\mu}) \rangle$ for all $\boldsymbol{\ell} \in \mathcal{Y}'_+$ and $\boldsymbol{\mu} \in \mathcal{X}_+$.

The sufficient conditions for strong duality and attainment of optimality between (48) and (49) as outlined in Theorem 2.6 of [29] are that:

- R1 All support sets are compact (A1)
- R2 All measure solutions have bounded mass (Lemma 3.2)
- R3 All functions involved in the definitions of c and $\mathcal{A}$ are continuous (A2, A3)
- R4 There exists a $\boldsymbol{\mu}_{\text{feas}} \in \mathcal{X}_+$ with $\mathbf{b} - \mathcal{A}(\boldsymbol{\mu}_{\text{feas}}) \in \mathcal{Y}_+$

The requirements R1 and R2 hold by Assumption A1 and Lemma 3.2 respectively. R3 is valid given that $c(x,y)$ is C^0 (A3), the projection map π^x is continuous, and the mapping $(t,x) \mapsto \mathcal{L}_f v(t,x)$ is C^0 for $v \in C^1$ and f Lipschitz (continuous) (A2). A feasible measure $\boldsymbol{\mu}_{\text{feas}}$ may be constructed from the process in Theorem 3.1 from a tuple $\mathcal{T}$, therefore satisfying R4.

Strong duality between (12) and (17) is therefore proven after satisfaction of all four requirements.

B Moment-SOS Hierarchy

The standard form for a measure LP with variable $\mu \in \mathcal{M}_+(X)$ involving a cost function $p \in C(X)$ and a (possibly infinite) set of affine constraints $\langle a_j, \mu \rangle = b_j$ with $b_j \in \mathbb{R}$ and $a_j \in C(X)$ for $j = 1, \ldots, J_{max}$ is,

$$p^* = \sup_{\mu \in \mathcal{M}_+(X)} \langle p, \mu \rangle \tag{50a}$$

$$\langle a_j(x), \mu \rangle = b_j \qquad \forall j = 1, \ldots, J_{max}. \tag{50b}$$

The dual problem to Program (50) with dual variables $v_j \in \mathbb{R} : \forall j = 1, \ldots, m$ is,

$$d^* = \inf_{v \in \mathbb{R}^m} \textstyle\sum_j b_j v_j \tag{51a}$$

$$p(x) - \textstyle\sum_j a_j(x) v_j \geq 0 \qquad \forall x \in X. \tag{51b}$$

The objectives in (50) and (51) will match ($p^* = d^*$ strong duality) if p^* is finite and if the mapping $\mu \to \{\langle a_j(x), \mu \rangle\}_{j=1}^m$ is closed in the weak-* topology (Theorem 3.10 in [38]).

When $p(x)$ and all $a_j(x)$ are polynomial, constraint (51b) is a polynomial nonnegativity constraint. The restriction that a polynomial $q(x) \in \mathbb{R}[x]$ is nonnegative over $\mathbb{R}^n$ may be strengthened to finding a set of polynomials $\{q_i(x)\}$ such that $q(x) = \sum_i q_i(x)^2$. The polynomials $\{q_i(x)\}$ are an SOS certificate of nonnegativity of $q(x)$, given that the square of a real quantity $q_i(x)$ at each i and x is nonnegative. The set of SOS polynomials in indeterminate quantities x is expressed as $\Sigma[x]$, with a maximal-degree-d subset of $\Sigma[x]_{\leq d}$.

The quadratic module $Q[g]$ formed by the constraints describing the basic semialgebraic set $\mathbb{K} = \{x \mid g_i(x) \geq 0, \ i = 1, \ldots, N_c\}$ is the set of polynomials:

$$Q[g] = \left\{ \sigma_0(x) + \textstyle\sum_{i=1}^{N_c} \sigma_i(x) g_i(x) \right\}, \tag{52}$$

such that the multipliers σ are SOS,

$$\sigma_i(x) \in \Sigma[x] \qquad \forall i = 0, \ldots, N_c. \tag{53}$$

The basic semialgebraic set $\mathbb{K}$ is compact if there exists a constant $0 \leq R < \infty$ such that $\mathbb{K}$ is contained in the ball $R \leq \|x\|_2^2$. $\mathbb{K}$ satisfies the Archimedean property if the polynomial $R - \|x\|_2^2$ is a member of $Q[g]$. The Archimedean property is stronger than compactness [39], and compact sets may be rendered Archimedean by adding a redundant ball constraint $R - \|x\|_2^2 \geq 0$ to the list of constraints describing in $\mathbb{K}$ (though finding such an R may be difficult). Putinar's Positivestellensatz gives necessary and sufficient conditions for a polynomial $p(x)$ to be positive over the Archimedean set $\mathbb{K}$: [40]:

$$\begin{aligned} & p(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) \\ & \sigma_0(x) \in \Sigma[x] \qquad \sigma_i(x) \in \Sigma[x]. \end{aligned} \tag{54}$$

The WSOS set $\Sigma[\mathbb{K}]$ is the set of polynomials that admit a positivity certificate over $\mathbb{K}$ from (54). Its maximal degree-d subset is $\Sigma[\mathbb{K}]_{\leq d}$).

Given a multi-index $\alpha \in \mathbb{N}^n$, the α-moment of a measure $\mu \in \mathcal{M}_+(X)$ is $\mathbf{m}_\alpha = \langle x^\alpha, \mu \rangle$. An infinite moment matrix $\mathbb{M}[\mathbf{m}]_{\alpha,\beta} = \mathbf{m}_{\alpha+\beta}$ indexed by monomials $\alpha, \beta \in \mathbb{N}^n$ may be constructed from the moment sequence $\mathbf{m}$.

The degree-d moment matrix $\mathbb{M}_d[\mathbf{m}]$, of size $\binom{n+d}{d}$ is the submatrix of $\mathbb{M}[\mathbf{m}]$ where the indices $\mathbb{M}_d[\mathbf{m}]_{\alpha,\beta}$ have total degree bounded by $0 \leq |\alpha|, |\beta| \leq d$. Given a polynomial $g(x) \in \mathbb{R}[x]$, the localizing matrix associated with g is a square infinite-dimensional symmetric matrix with entries $\mathbb{M}[g\mathbf{m}]_{\alpha,\beta} = \sum_{\gamma \in \mathbb{N}^n} g_\gamma \mathbf{m}_{\alpha+\beta+\gamma}$. A moment sequence m has a representing measure $\mu \in \mathcal{M}_+(\mathbb{K})$ if there exists μ supported in $\mathbb{K}$ such that $\mathbf{m}_\alpha = \langle x^\alpha, \mu \rangle \ \forall \alpha \in \mathbb{N}^n$. The LMI conditions that $\mathbb{M}[\mathbf{m}] \succeq 0$ and $\mathbb{M}[g_i \mathbf{m}] \succeq 0 \ \forall i = 1, \ldots, N_c$ are necessary to guarantee the existence of a representing measure associated with $\mathbf{m}$. The moment matrix $\mathbb{M}[\mathbf{m}]$ is a localizing matrix with the function $g = 1$. These LMI conditions are sufficient if the set $\mathbb{K}$ is Archimedean, and all compact sets may be rendered Archimedean through the application of a redundant ball constraint [40].

Assume that each polynomial $g_i(x)$ in the constraints of $\mathbb{K}$ has a degree d_i. We define a block-diagonal matrix $\mathbb{M}_d[\mathbb{K}\mathbf{m}]$ containing the moment and all localizing matrices as,

$$\text{diag}(\mathbb{M}_d[\mathbf{m}], \{\mathbb{M}_{d-d_i}(g_i\mathbf{m})\ \forall i = 1, \ldots, N_c\}). \tag{55}$$

The degree-d moment relaxation of Problem (50) with variables $y \in \mathbb{R}^{\binom{n+2d}{2d}}$ is,

$$p_d^* = \max_{\mathbf{m}} \textstyle\sum_\alpha p_\alpha \mathbf{m}_\alpha, \qquad \mathbb{M}_d[\mathbb{K}\mathbf{m}] \succeq 0 \tag{56a}$$

$$\textstyle\sum_\alpha a_{j\alpha}\mathbf{m}_\alpha \ = b_j \qquad \forall j = 1, \ldots, m. \tag{56b}$$

The bound $p_d^* \geq p^*$ is an upper bound for the infinite-dimensional measure LP. The decreasing sequence of upper bounds $p_d^* \geq p_{d+1}^* \geq \ldots \geq p^*$ is convergent to p^* as $d \to \infty$ if $\mathbb{K}$ is Archimedean. The dual semidefinite program to (56a) is the degree-d SOS relaxation of (51):

$$d_d^* = \min_{v \in \mathbb{R}^m} \textstyle\sum_j b_j v_j \tag{57a}$$

$$p(x) - \textstyle\sum_j a_j(x) v_j = \sigma_0(x) + \sum_k \sigma_i(x) g_i(x) \tag{57b}$$

$$\sigma(x) \in \Sigma[x]_{\leq 2d} \tag{57c}$$

$$\sigma_i(x) \in \Sigma[x]_{\leq 2d - \lceil \deg g_i/2 \rceil} \quad \forall i \in 1, \ldots, N_c. \tag{57d}$$

We use the convention that the degree-d SOS tightening of (57) involves polynomials of maximal degree $2d$. When the moment sequence $\mathbf{m}_\alpha$ is bounded ($|\mathbf{m}_\alpha| < \infty\ \forall |\alpha| \leq 2d$) and there exists an interior point of the affine measure constraints in (50b), then the finite-dimensional truncations (56a) and (57) will also satisfy strong duality $p_k^* = d_k^*$ at each degree k (by arguments from Appendix D/Theorem 4 of [11] using Theorem 5 of [41], also applied in Corrolary 8 of [21]). The sequence of upper bounds (outer approximations) $p_d^* \geq p_{d+1}^* \geq \ldots$ computed from LMIs is called the Moment-SOS hierarchy.

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