Peak Estimation for Time-Delay Systems

Jared Miller
Milan Korda
Victor Magron
Mario Sznaier
MERIIT Lab (NYU)
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Main Ideas

Time delays are hard

Define measure-valued solutions

Form upper-bounds to peak values

Time-Delay Background

Time-Delay Examples

Delay between state change and its effect on system

System	**Delay**
Epidemic	Incubation Period
Population	Gestation Time
Traffic	Reaction Time
Congestion	Queue Time
Fluid Flow	Moving in Pipe

Functional Differential Equation

Dynamics Model

Finite number of bounded discrete delays

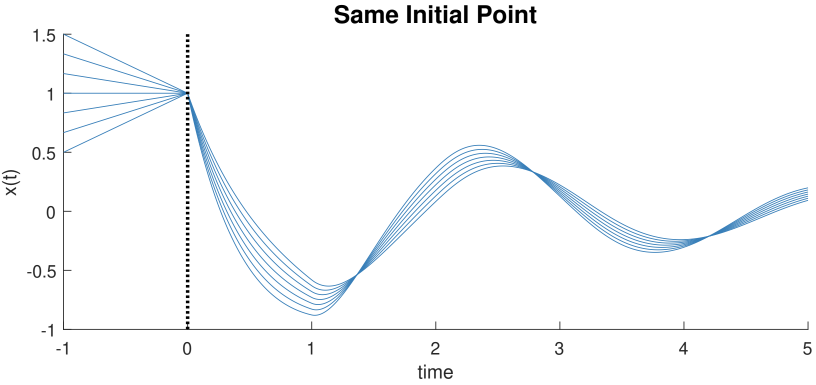
$$0 = \tau_0 < \tau_1 < \ldots < \tau_r < \infty$$

Dynamics for initial history $x_h(t)$

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau_1), x(t-\tau_2),\ \ldots,\ x(t-\tau_r)) \\ x(t) &= x_h(t) & \forall t \in [-\tau_r, 0] \end{aligned}$$

History $x_h(t)$ does not have to obey dynamics

Dependence on History

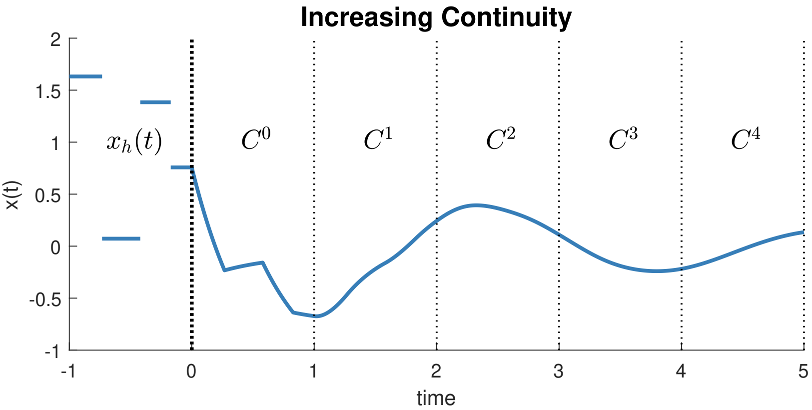


$$x'(t) = -2x(t) - 2x(t-1)$$

All trajectories pass through $(t, x) = (0, 1)$

Initial history determines behavior, not just initial point

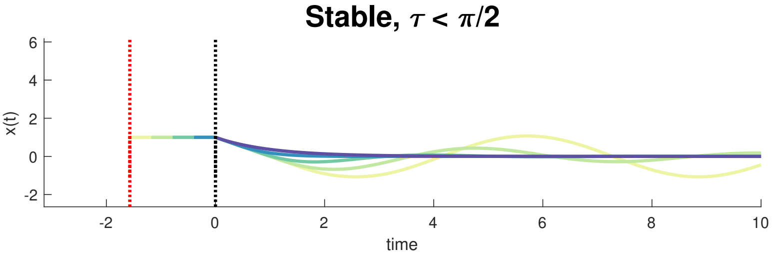
Propagation of Continuity

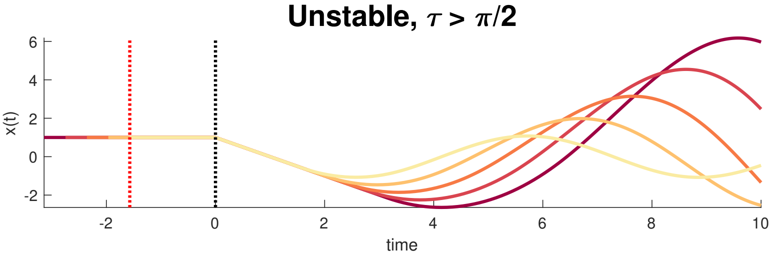


$$x'(t) = -2x(t) - 2x(t-1)$$

Continuity increases every τ_r time steps

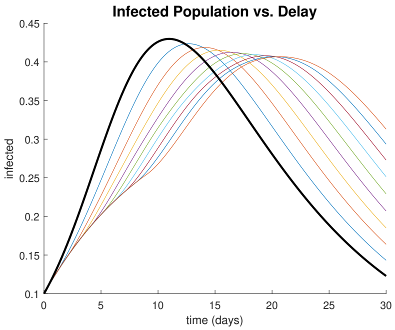
Delay Bifurcation Example



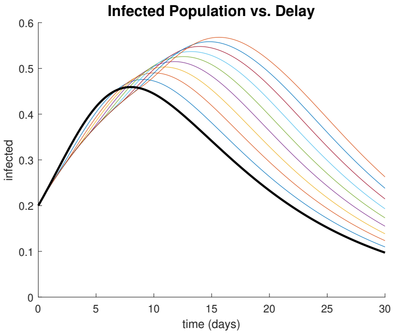


$$x'(t) = -x(t - \tau) \qquad \text{(Fridman 2014)}$$

Peak Value vs. Delay



(a) $I_h = 0.1$, peak decreases



(b) $I_h = 0.2$, peak increases

$$\begin{bmatrix} S'(t) \\ I'(t) \end{bmatrix} = \begin{bmatrix} -0.4S(t)I(t) \\ 0.4S(t-\tau)I(t-\tau) - 0.1I(t) \end{bmatrix}$$

Existing Methods (very brief)

Certificates of Stability

- Lyapunov-Krasovskii
- Razumikhin
- Hanalay
- ODE-Transport PDE

Relaxed control (Warga 1974, Vinter and Rosenblueth 1991-2)

SOS Barrier (Papachristodoulou and Peet, 2010)

Fixed-terminal-time OCP with gridding (Barati 2012)

Riesz Operators (Magron and Prieur, 2020)

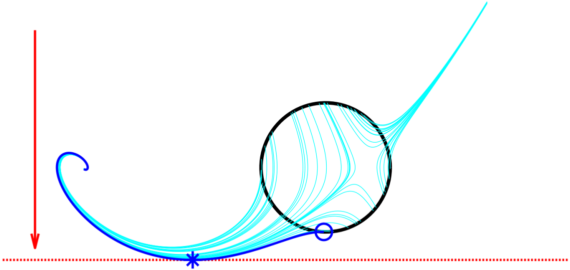
Peak Estimation (ODE)

Peak Estimation Background

Find supremal value of $p(x)$ along ODE trajectories

$$P^* = \sup_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0.$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

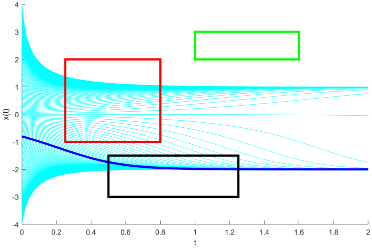
Occupation Measures

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

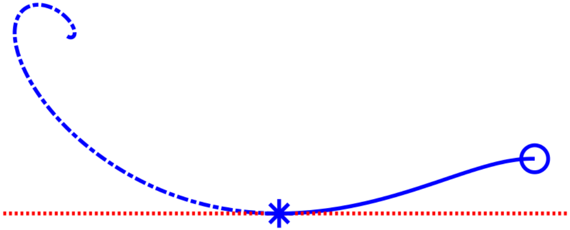
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x + 2)(x - 1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\begin{aligned}
\mu_0^* : \quad & \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : \quad & \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : \quad & \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*)) dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

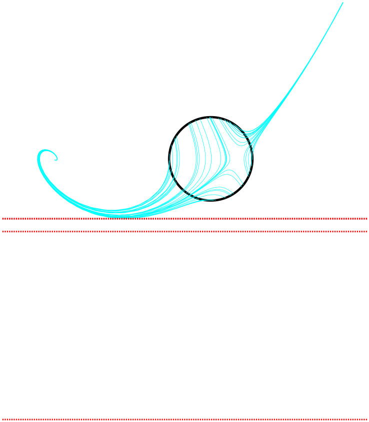
$$
\begin{aligned}
p^* = \ \sup \quad & \langle p(x), \mu_p \rangle & \text{(1a)}\\
& \langle 1, \mu_0 \rangle = 1 & \text{(1b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v & \text{(1c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) & \text{(1d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) & \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Peak Estimation (Delayed)

Peak Estimation

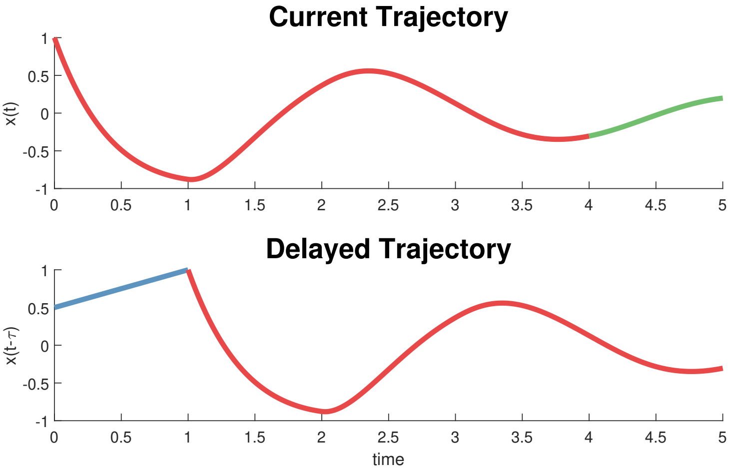
History $x_h(t)$ resides in a class of functions $\mathcal{H}$

Graph-constrained $\mathcal{H}$: $(t, x_h(t))$ contained in $H_0 \subset [-\tau, 0] \times X$

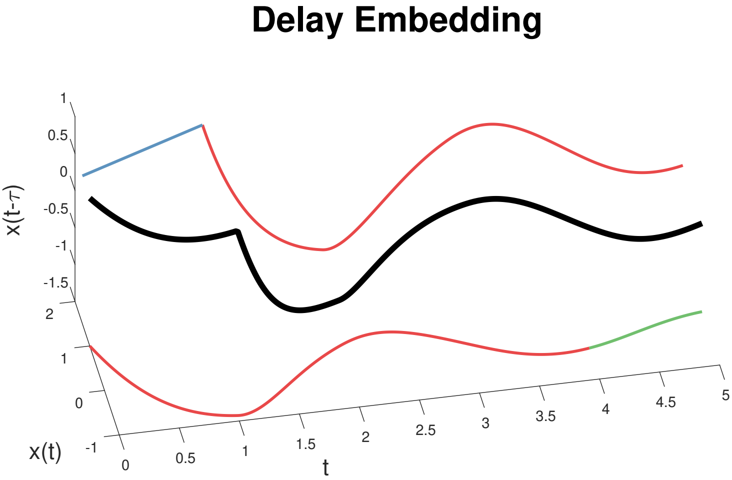
$$\begin{aligned}
P^* = \sup_{t^*,\, x_h} \ & p(x(t^*)) \\
& \dot{x} = f(t, x(t), x(t-\tau)) && t \in [0, t^*] \\
& x(t) = x_h(t) && t \in [-\tau, 0] \\
& x_h(\cdot) \in \mathcal{H}
\end{aligned}$$

Represent $x(t \mid x_h) : t \in [-\tau, t^*]$ as occupation measure

Time-Delay Visualization



$$x(t) = -2x(t) - 2x(t-1), \qquad x_h(t) = 1 - t/2$$



Black curve: $(t, x(t), x(t-\tau))$

Measure-Valued Solution

Tuple of measures for the delayed case

History	$\mu_{\boldsymbol{h}} \in \mathcal{M}_+(H_0)$
Initial	$\mu_0 \in \mathcal{M}_+(X_0)$
Peak	$\mu_{\boldsymbol{p}} \in \mathcal{M}_+([0, T] \times X)$
Occupation Start	$\bar{\mu}_0 \in \mathcal{M}_+([0, T-\tau] \times X^2)$
Occupation End	$\bar{\mu}_1 \in \mathcal{M}_+([T-\tau, T] \times X^2)$
Time-Slack	$\nu \in \mathcal{M}_+([0, T] \times X)$

Types of Constraints

History-Validity: initial conditions

Liouville: Dynamics

Consistency: Time-delay overlaps

History Validity

History $(t, x_h(t))$ defines a curve $[-\tau, 0]$, point at $x_h(0)$

Point evaluation $\langle 1, \mu_0 \rangle = 0$

t-marginal of μ_h should be the Lebesgue measure in $[-\tau, 0]$

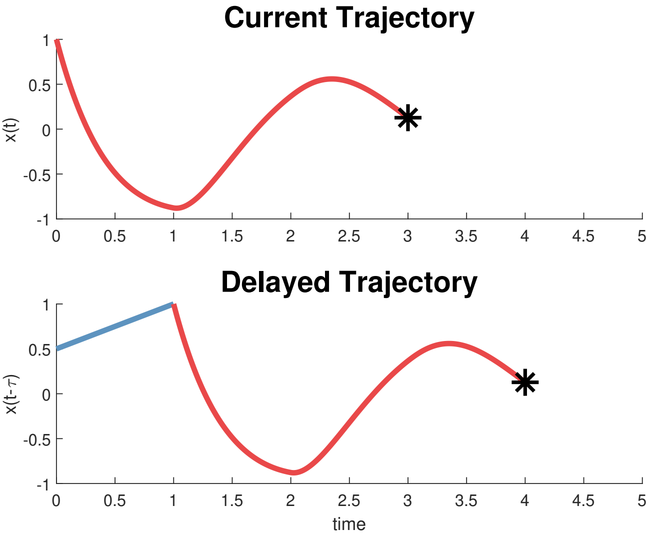
Liouville

Sum $\bar{\mu} = \bar{\mu}_0 + \bar{\mu}_1$ is a relaxed occupation measure of the delay embedding $(t, x(t), x(t-\tau))$

For all test functions $v \in C^1([0, T] \times X)$:

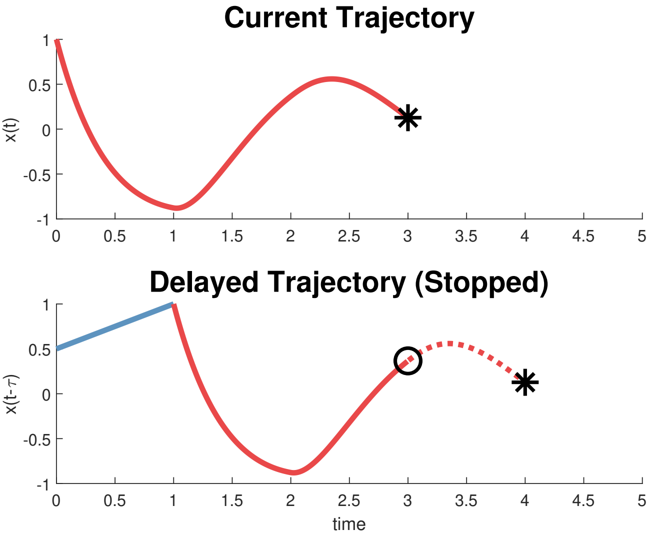
$$\langle v, \mu_p \rangle = \langle v(0, x), \mu_0(x) \rangle + \langle v(t, x_0), \bar{\mu}_0(t, x_0, x_1) + \bar{\mu}_1(t, x_0, x_1) \rangle$$

Consistency Issue



Inconsistent elapsed times

Consistency Fix



Early stopping in delayed time

Consistency Constraint

Inspired by changing limits of integrals

$$\left(\int_0^{t^*} + \int_{t^*}^{\min(T, t^*+\tau)}\right)\phi(t, x(t-\tau))dt$$
$$= \left(\int_{-\tau}^{0} + \int_0^{\min(t^*, T-\tau)}\right)\phi(t'+\tau, x(t'))dt'.$$

Shift-push $S^\tau_\#$ with $\langle \phi, S^\tau_\# \mu\rangle = \langle S^\tau \phi, \mu\rangle = \langle \phi(t+\tau, x), \mu\rangle$

Consistency constraint with time-slack ν

$$\pi^{tx_1}_\#(\bar{\mu}_0 + \bar{\mu}_1) + \nu = S^\tau_\#(\mu_h + \pi^{tx_0}_\# \bar{\mu}_0).$$

Measure Linear Program

Linear program for time-delay peak estimation

$$
\begin{aligned}
p^* = \ \sup \quad & \langle p, \mu_p \rangle && \text{(2a)}\\
& \text{History-Validity}(\mu_0, \mu_h) && \text{(2b)}\\
& \text{Liouville}(\mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1) && \text{(2c)}\\
& \text{Consistency}(\bar{\mu}_h, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2d)}\\
& \text{Measure Definitions for } (\mu_h, \mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2e)}
\end{aligned}
$$

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lfloor \deg f/2 \rfloor$

Bounds: $p^*_d \geq p^*_{d+1} \geq \ldots = p^* \geq P^*$

Size of Moment Matrices Peak Estimation

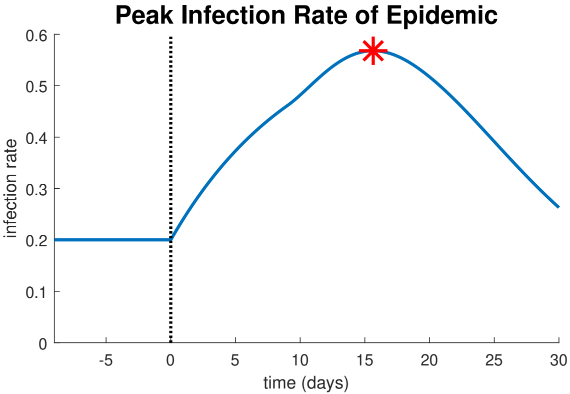
Measure:	μ_0	μ^p	μ_h
Size:	$\binom{n+d}{d}$	$\binom{n+1+d}{d}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Measure:	$\bar{\mu}_0$	$\bar{\mu}_1$	ν
Size:	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(2n+1)^{6\tilde{d}}$ or $\tilde{d}^{4(2n+1)}$

Examples

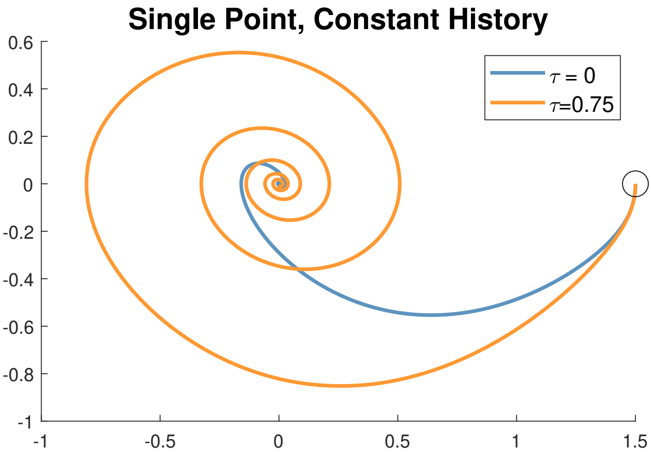
SIR Peak Estimation Example



Upper bound $I_{max} \geq 56.9\%$ with order 3 LMI

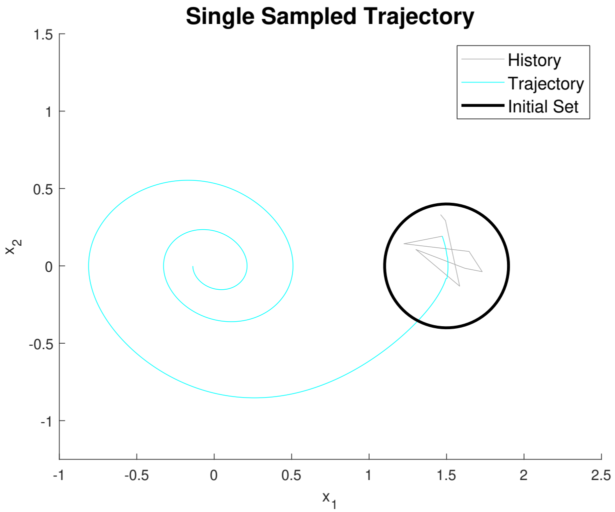
Recovery: $t_* = 15.6$ days, $(S^*, I^*) = (56.9\%, 5.61\%)$

Delay Comparision



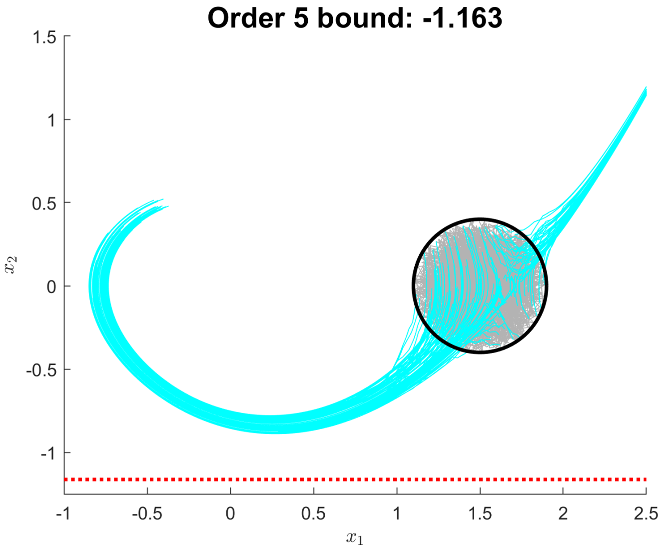
$$\dot{x}(t) = \begin{bmatrix} x_2 \\ -x_1(t-\tau) - x_2(t) + x_1(t)^3/3 \end{bmatrix}$$

Single History Plot



Random history in $\{x \mid (x_1 - 1.5)^2 + x_2^2 \leq 0.4^2\}$

Peak Estimate with Multiple Histories



Minimize x_2 on the delayed Flow system

Distance Estimate with Multiple Histories

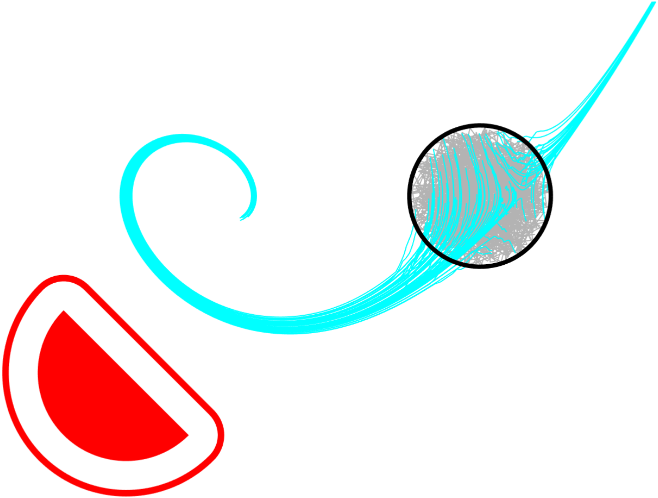
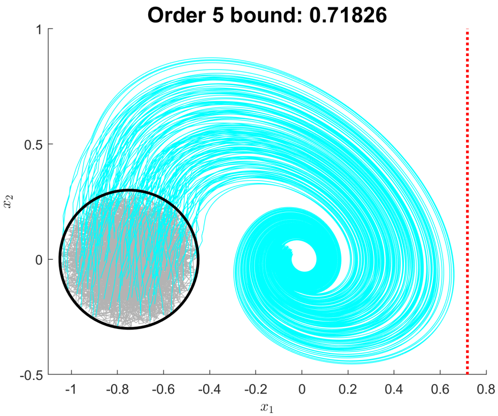


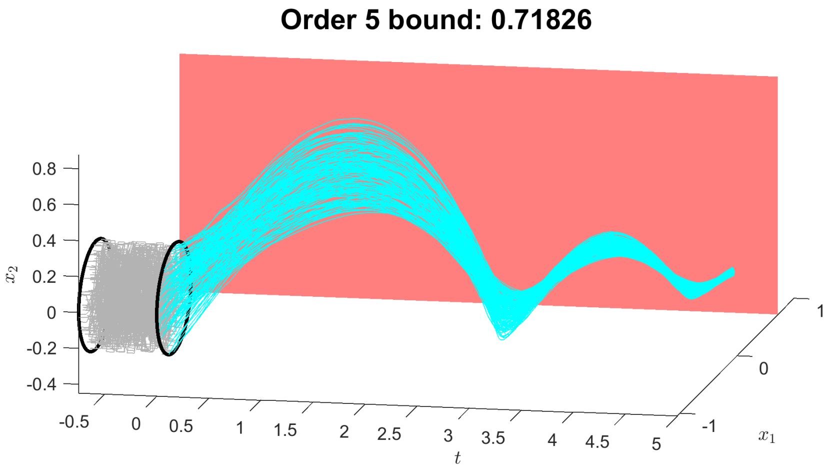
Figure 2: Minimize $c(x; X_u)$ on the delayed Flow system

Time-Varying System



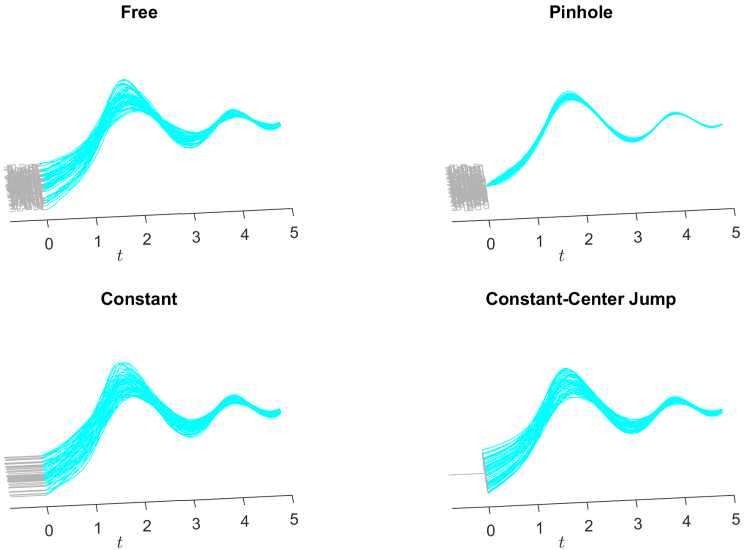
Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Time-Varying System (Cont.)



3d view of system

Time-Varying Histories



History restrictions and trajectories of system

Take-aways

Conclusion

Posed peak estimation problem for delayed system

Defined measure-valued solutions

Solved sequence of SDPs to get peak bounds

Future Work

- Conditions for no conservatism
- Improve scaling/computational complexity
- Better bounds and conditioning
- Other delay structures (e.g. discrete-time, proportional)
- Reachable set estimation

Thank you for your attention