Peak Estimation of Time Delay Systems Using Occupation Measures

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Time-Delay Examples

Delay between state change and its effect on system

System	Delay
Epidemic	Incubation Period
Population	Gestation Time
Traffic	Reaction Time
Congestion	Queue Time
Fluid Flow	Moving in Pipe

Modeled as a functional differential equation

Flow of Presentation

Formulate an ODE Peak estimation

Solve using infinite-dimensional LP

Adapt ODE LP mehtod to time-delays

Truncate using polynomial optimization (moment-SOS)

Watch out for hazards (conservatism)

Dynamics Model

Delay Differential Equation (DDE) for history $x_h(t)$

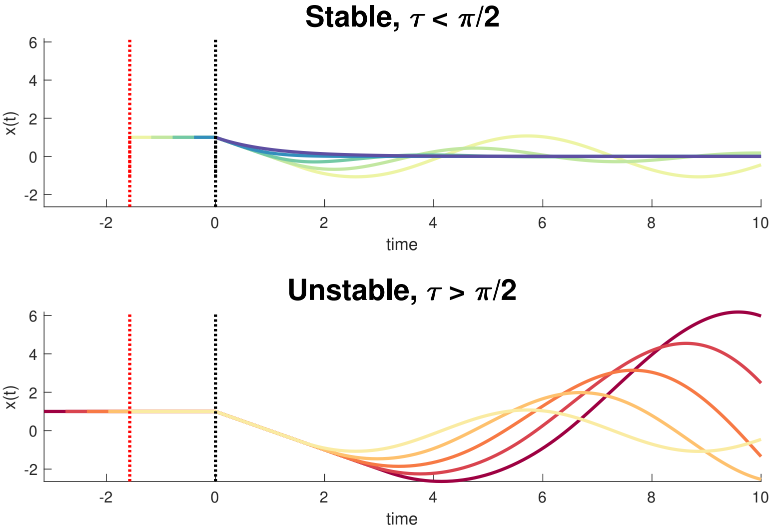
$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau)) \\ x(t) &= x_h(t) \qquad \forall t \in [-\tau, 0] \end{aligned}$$

History $x_h(t)$ does not have to obey dynamics

Can be extended to multiple delays $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_r$

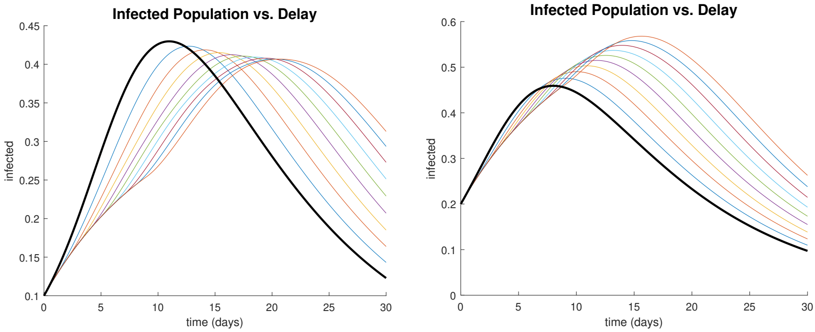
Others: proportional $x(\kappa t)$, distributed $\int_{-\tau_r}^{0} g(\tau')x(t+\tau')d\tau'$

Delay Bifurcation Example



$x'(t) = -x(t - \tau)$ (Fridman 2014)

Peak Value vs. Delay



(a) $I_h = 0.1$, peak decreases **(b)** $I_h = 0.2$, peak increases

$$\begin{bmatrix} S'(t) \\ I'(t) \end{bmatrix} = \begin{bmatrix} -0.4S(t)I(t) \\ 0.4S(t-\tau)I(t-\tau) - 0.1I(t) \end{bmatrix}$$

Existing Methods (very brief)

Certificates of Stability

- Lyapunov-Krasovskii
- Razumikhin
- Hanalay
- ODE-Transport PDE

Relaxed control (Warga 1974, Vinter and Rosenblueth 1991-2)

SOS Barrier (Papachristodoulou and Peet, 2010)

Fixed-terminal-time OCP with gridding (Barati 2012)

Riesz Operators (Magron and Prieur, 2020)

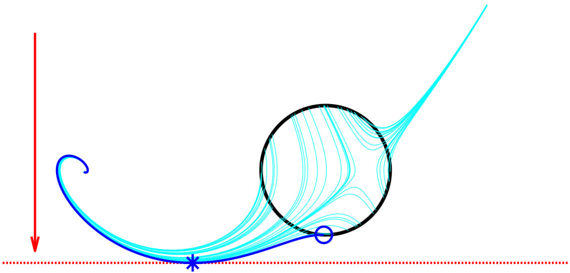
Peak Estimation (ODE)

Peak Estimation Background

Find supremal value of $p(x)$ along ODE trajectories

$$P^* = \sup_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0.$$



$$p(x) = -x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

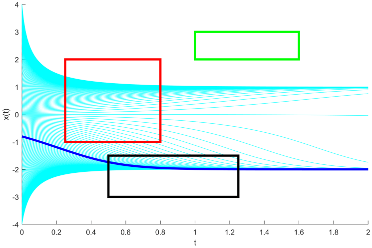
Occupation Measures

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x + 2)(x - 1)$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

$$p^* = \sup \quad \langle p(x), \mu_p \rangle \tag{1a}$$

$$\langle 1, \mu_0 \rangle = 1 \tag{1b}$$

$$\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v \tag{1c}$$

$$\mu, \mu_p \in \mathcal{M}_+([0, T] \times X) \tag{1d}$$

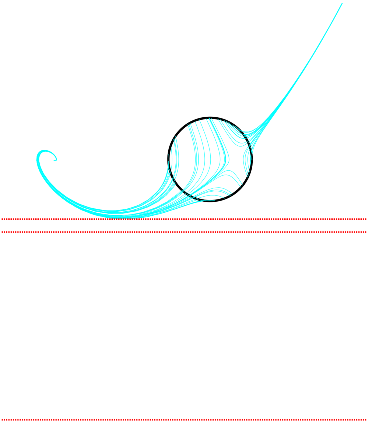
$$\mu_0 \in \mathcal{M}_+(X_0) \tag{1e}$$

Test functions $v(t,x) \in C^1([0, T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Peak Estimation (Delayed)

Peak Estimation

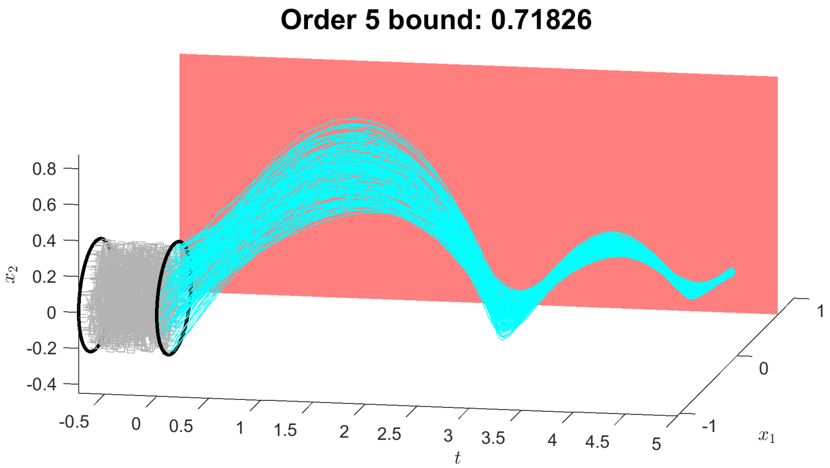
History $x_h(t)$ resides in a class of functions $\mathcal{H}$

Graph-constrained $\mathcal{H} : (t, x_h(t))$ contained in $H_0 \subset [-\tau, 0] \times X$

$$
\begin{aligned}
P^* = \sup_{t^*,\, x_h} \; & p(x(t^*)) \\
& \dot{x} = f(t, x(t), x(t-\tau)) && t \in [0, t^*] \\
& x(t) = x_h(t) && t \in [-\tau, 0] \\
& x_h(\cdot) \in \mathcal{H}
\end{aligned}
$$

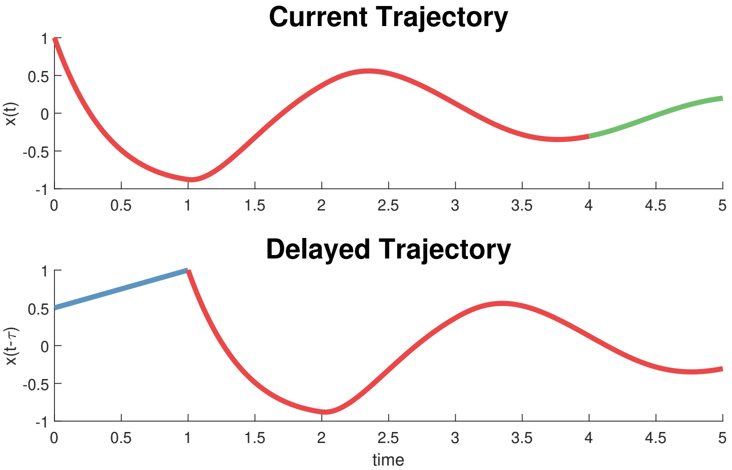
Represent $x(t \mid x_h) : t \in [-\tau, t^*]$ as occupation measure

Time-Varying System



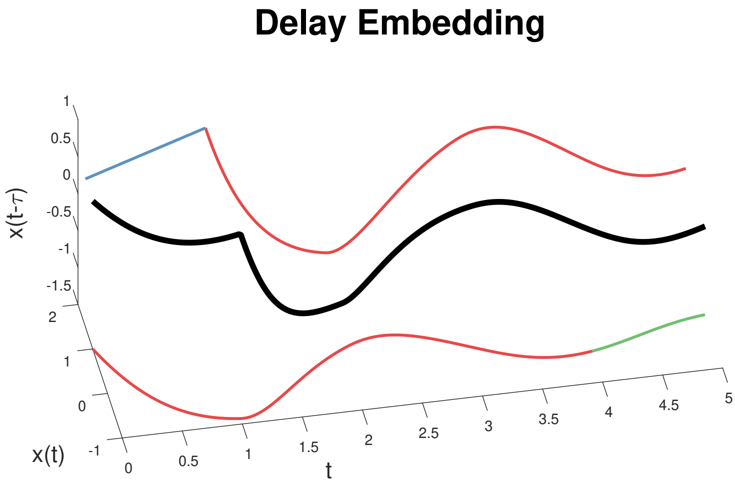
Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Time-Delay Visualization



$$x(t) = -2x(t) - 2x(t-1), \qquad x_h(t) = 1 - t/2$$

Time-Delay Embedding



Black curve: $(t, x(t), x(t-\tau))$

Measure-Valued Solution

Tuple of measures for the delayed case

History	$\mu_{\boldsymbol{h}} \in \mathcal{M}_+(H_0)$
Initial	$\mu_0 \in \mathcal{M}_+(X_0)$
Peak	$\mu_{\boldsymbol{p}} \in \mathcal{M}_+([0, T] \times X)$
Occupation Start	$\bar{\mu}_0 \in \mathcal{M}_+([0, T - \tau] \times X^2)$
Occupation End	$\bar{\mu}_1 \in \mathcal{M}_+([T - \tau, T] \times X^2)$
Time-Slack	$\nu \in \mathcal{M}_+([0, T] \times X)$

Types of Constraints

History-Validity: initial conditions

Liouville: Dynamics

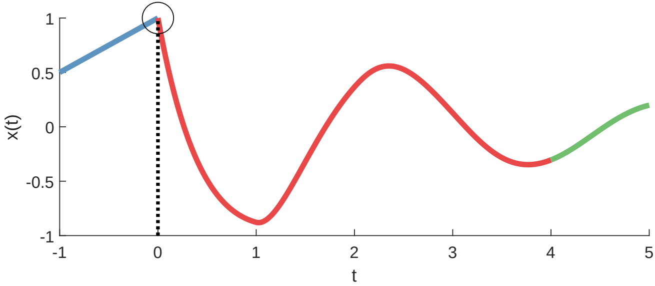
Consistency: Time-delay overlaps

History Validity

History $(t, x_h(t))$ defines a curve $[-\tau, 0]$, point at $x_h(0)$

Point evaluation $\langle 1, \mu_0 \rangle = 0$

t-marginal of μ_h should be the Lebesgue measure in $[-\tau, 0]$



History and Initial ∘

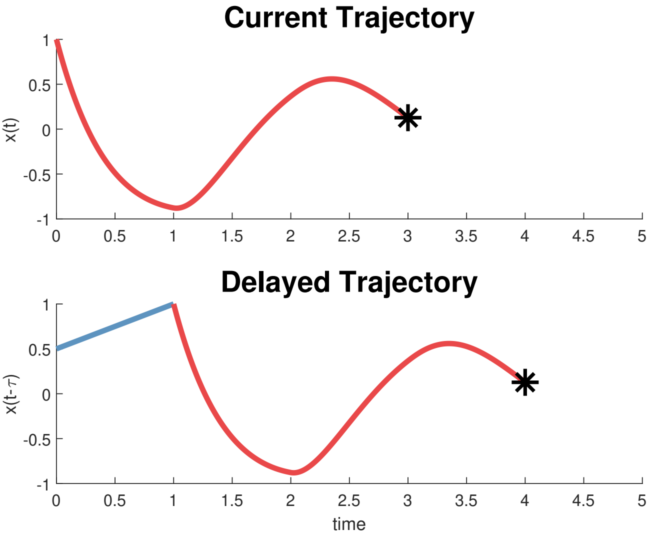
Liouville

Sum $\bar{\mu} = \bar{\mu}_0 + \bar{\mu}_1$ is a relaxed occupation measure of the delay embedding $(t, x(t), x(t-\tau))$

For all test functions $v \in C^1([0, T] \times X)$:

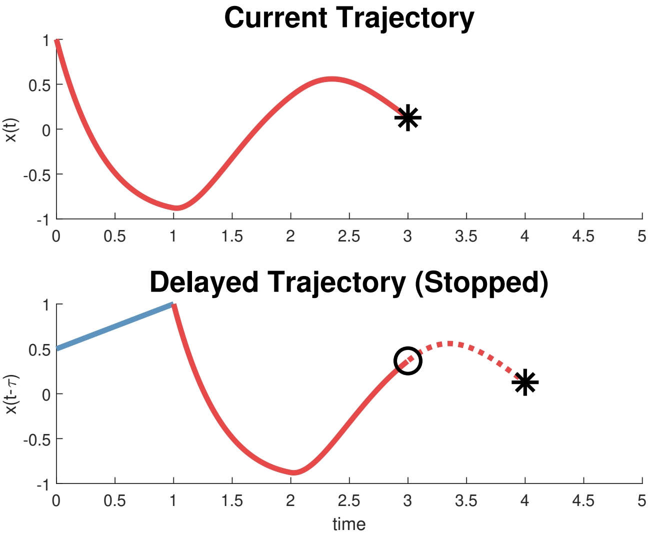
$$\langle v, \mu_p \rangle = \langle v(0, x), \mu_0(x) \rangle + \langle \mathcal{L}_f v, \bar{\mu}_0 + \bar{\mu}_1 \rangle$$

Consistency Issue



Inconsistent elapsed times

Consistency Fix



Early stopping in delayed time, add slack measure ν

Measure Linear Program

Linear program for time-delay peak estimation

$$
\begin{aligned}
p^* = \sup \quad & \langle p, \mu_p \rangle && \text{(2a)}\\
& \text{History-Validity}(\mu_0, \mu_h) && \text{(2b)}\\
& \text{Liouville}(\mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1) && \text{(2c)}\\
& \text{Consistency}(\bar{\mu}_h, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2d)}\\
& \text{Measure Definitions for } (\mu_h, \mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2e)}
\end{aligned}
$$

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lfloor \deg f/2 \rfloor$

Bounds: $p^*_d \geq p^*_{d+1} \geq \ldots = p^* \geq P^*$

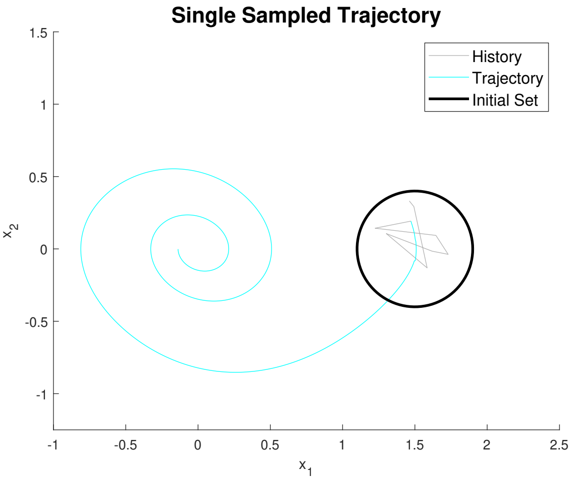
Size of Moment Matrices Peak Estimation

Measure:	μ_0	μ^p	μ_h
Size:	$\binom{n+d}{d}$	$\binom{n+1+d}{d}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$
Measure:	$\bar{\mu}_0$	$\bar{\mu}_1$	ν
Size:	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(2n+1)^{6\tilde{d}}$ or $\tilde{d}^{4(2n+1)}$

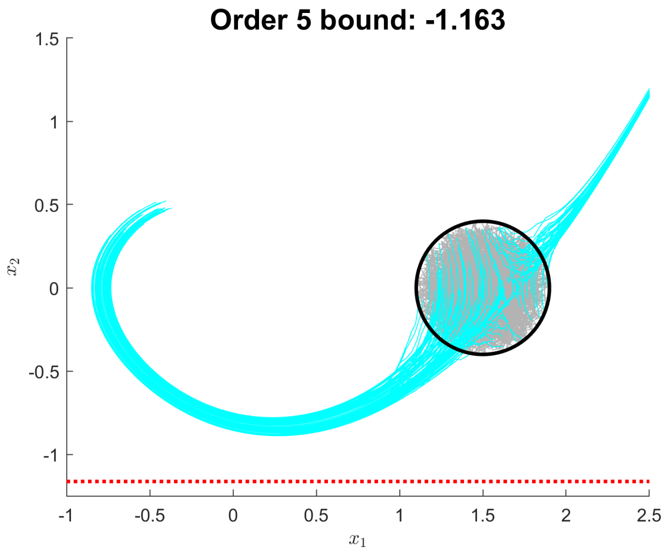
Examples

Single History Plot



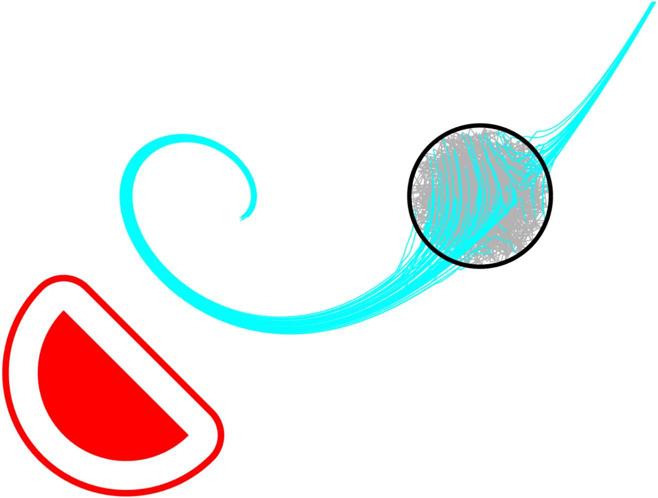
$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -x_1(t-\tau) - x_2(t) + x_1(t)^3/3 \end{bmatrix}$$

Peak Estimate with Multiple Histories



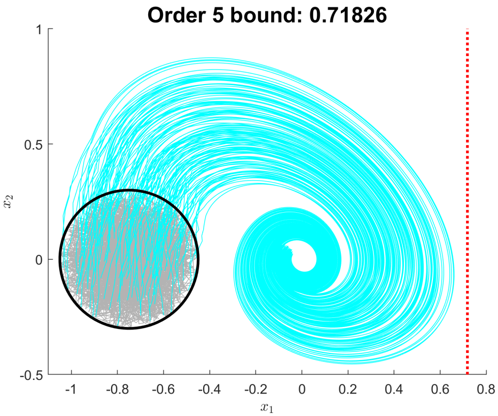
Minimize x_2 on the delayed Flow system

Distance Estimate with Multiple Histories



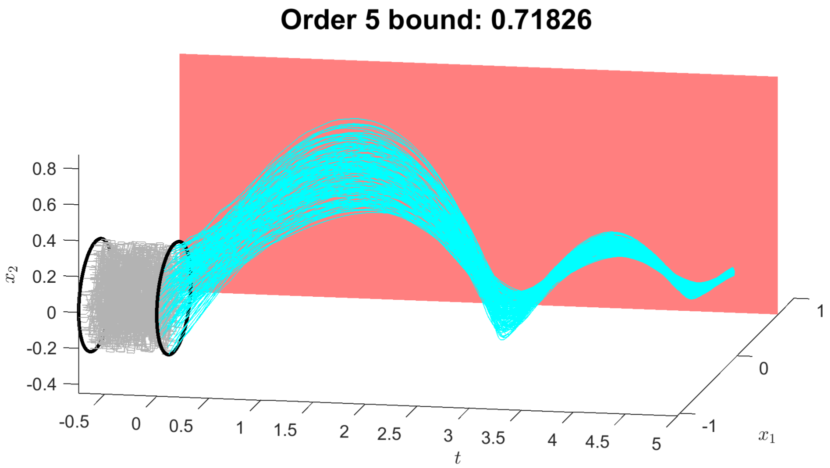
Minimize $c(x; X_u)$ on the delayed Flow system

Time-Varying System



Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Time-Varying System (Cont.)



3d view of system

Take-aways

Conclusion

Posed peak estimation problem for delayed system

Defined measure-valued solutions

Solved sequence of SDPs to get peak bounds

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Thanks!

Questions?

Bonus Slides

Consistency Constraint

Inspired by changing limits of integrals

$$\left(\int_0^{t^*} + \int_{t^*}^{\min(T, t^*+\tau)}\right) \phi(t, x(t-\tau))dt$$
$$= \left(\int_{-\tau}^{0} + \int_{0}^{\min(t^*, T-\tau)}\right) \phi(t'+\tau, x(t'))dt'.$$

Shift-push $S^\tau_\#$ with $\langle \phi, S^\tau_\# \mu\rangle = \langle S^\tau \phi, \mu\rangle = \langle \phi(t+\tau, x), \mu\rangle$

Consistency constraint with time-slack ν

$$\pi^{tx_1}_\#(\bar{\mu}_0 + \bar{\mu}_1) + \nu = S^\tau_\#(\mu_h + \pi^{tx_0}_\# \bar{\mu}_0).$$