Safety Quantification for Nonlinear and Time-Delay Systems using Occupation Measures

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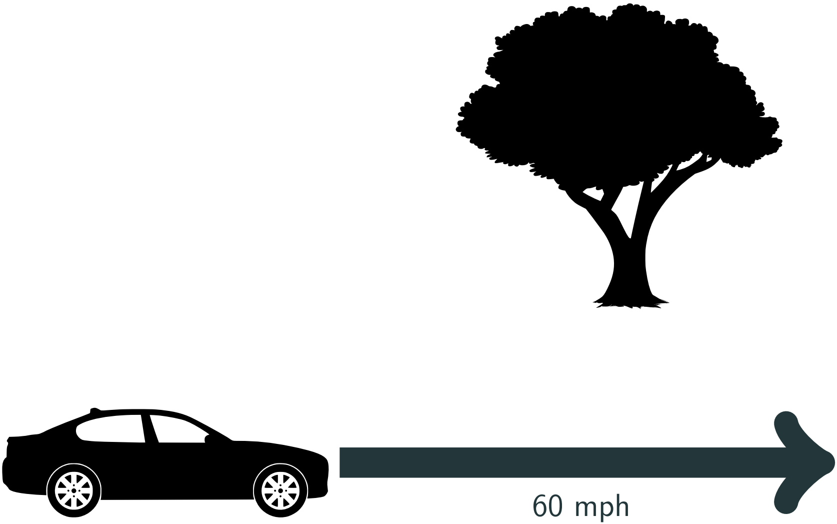
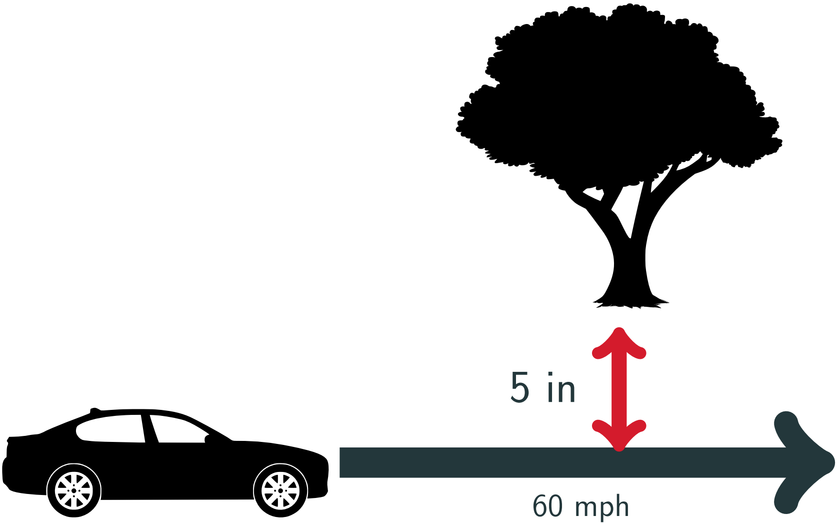
Safety Example



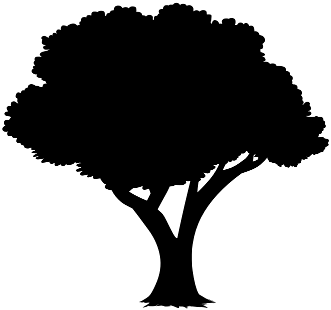
Safety Example (Barrier/Density Function)

Safety Example (Distance Estimate)



Safety Example

Safety Quantification



Motivation: Epidemic

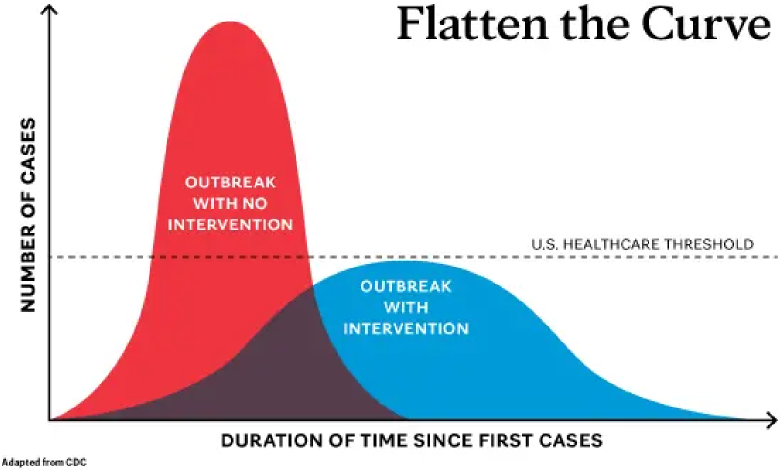
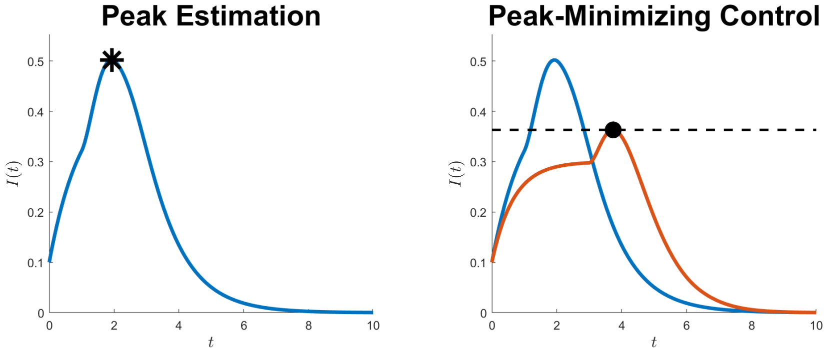


Image credit to Mayo Clinic News Network

Problems Covered



Main Ideas

Pose safety quantification problems

Want convex, convergent, bisection-free algorithms

Formulate using convex linear programs in measures

Increasing-quality bounds using Semidefinite Programming

Overview of Presentation

Peak estimation background

1. Survey of Thesis Work
2. Peak Value-at-Risk Estimation
3. Time-Delay Systems

Wrap-up

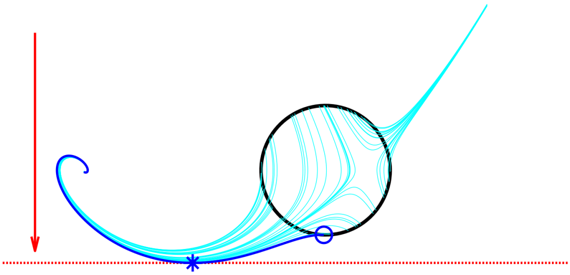
Peak Estimation Background

Peak Estimation Background

Find extreme value of $p(x)$ along trajectories

$$P^* = \sup_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0.$$



$$p(x) = -x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

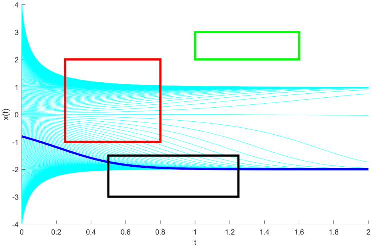
Occupation Measure

Time trajectories spend in set

Test function
$v(t,x) \in C([0,T] \times X)$

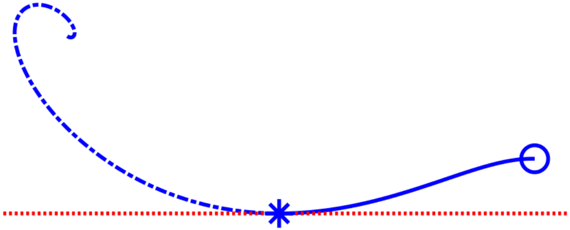
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t,x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\begin{aligned}
\mu_0^* : &\qquad \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : &\qquad \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : &\qquad \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*)) dt
\end{aligned}$$

Liouville Equation

Lie derivative (instantaneous change along f) $\forall v \in C^1$:

$$\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x) \tag{1a}$$

Conservation law: final = initial + accumulated change

$$\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \tag{1b}$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{1c}$$

Liouville 'represents' dynamics $\dot{x}(t) = f(t, x(t))$

Measures for Peak Estimation

Infinite-dimensional Linear Program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \ \sup \quad & \langle p(x), \mu_p \rangle && \text{(2a)}\\
& \langle 1, \mu_0 \rangle = 1 && \text{(2b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(2c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(2d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(2e)}
\end{aligned}
$$

Instance of Optimal Control Program (Lewis and Vinter, 1980)

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle \leq p^*$

$P^* = p^*$ if compactness, Lipschitz properties hold

Moments for Peak Estimation

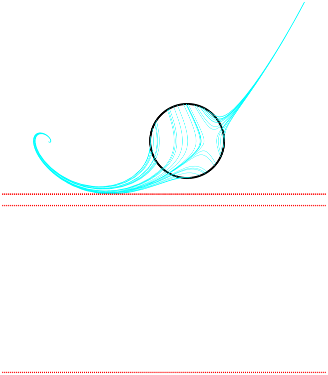
Moment: $y_\alpha = \langle x^\alpha, \nu \rangle \ \forall \alpha \in \mathbb{N}^n$

Moment matrix $\mathbb{M}[y]_{\alpha\beta} = y_{\alpha+\beta}$ is PSD

$$\mathbb{M}_2[y] = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{11} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0$$

Liouville induces affine relation in $(\mu^0, \mu^p, \mu) \to (y^0, y^p, y)$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)

Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Max. PSD size: $\binom{(n+1)+(d+\lfloor \deg f/2 \rfloor)}{n+1}$ (Fantuzzi, Goluskin, 2020)

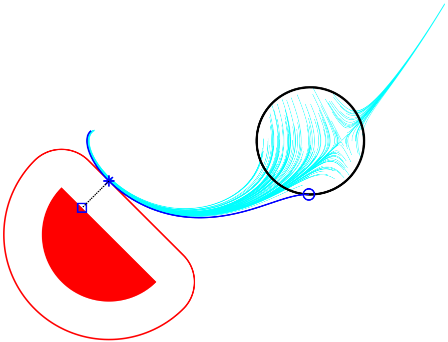
Survey of Thesis Work

Distance Estimation Problem

Unsafe set X_u, point-set distance $c(x; X_u) = \inf_{y \in X_u} c(x, y)$

$$P^* = \inf_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \quad x(0) = x_0.$$



L_2 bound of 0.2831

Distance Program (Measures)

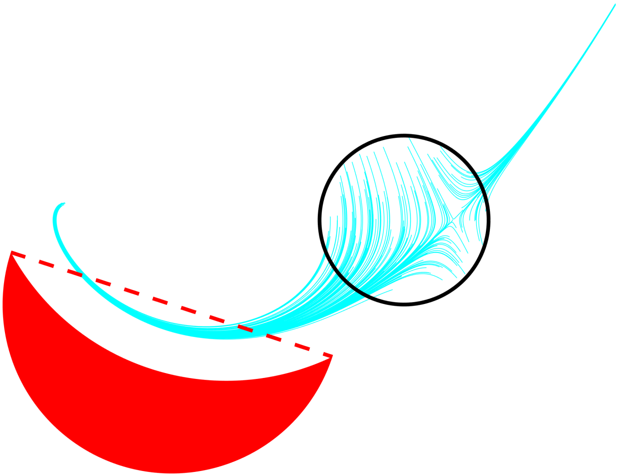
Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \inf \quad & \langle c(x,y), \eta(x,y) \rangle && && (3a)\\
& \langle 1, \mu_0 \rangle = 1 && && (3b)\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle && \forall v && (3c)\\
& \langle w(x), \eta(x,y) \rangle = \langle w(x), \mu_p(t,x) \rangle && \forall w && (3d)\\
& \eta \in \mathcal{M}_+(X \times X_u) && && (3e)\\
& \mu_p, \ \mu \in \mathcal{M}_+([0,T] \times X) && && (3f)\\
& \mu_0 \in \mathcal{M}_+(X_0) && && (3g)
\end{aligned}
$$

Probability measures: (μ_0, μ_p, η)

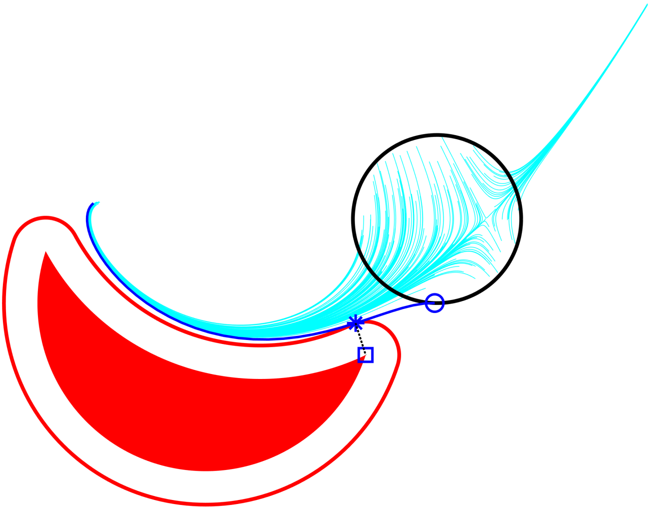
Near-optimal trajectories if moment-matrix $\approx$ rank-1

Distance Example (Flow Moon)



Collision if X_u was a half-circle

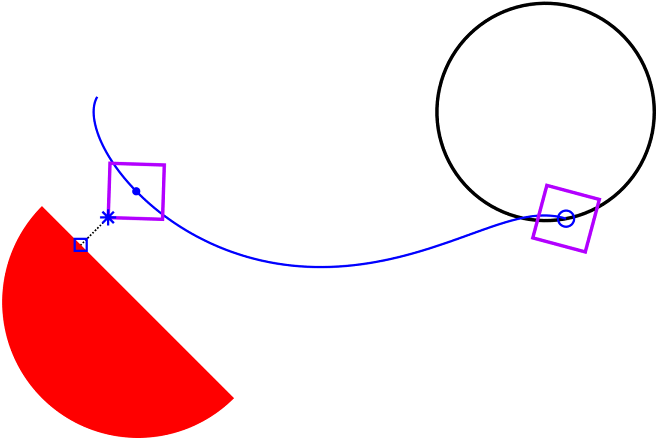
Distance Example (Flow Moon)



L_2 bound of 0.1592

Safety of Shapes

Points on shape S with orientation ω (e.g., rigid body motion)

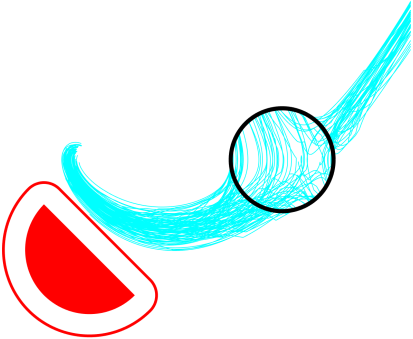


L_2 bound of 0.1465, rotating square

Distance with Bounded Uncertainty

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$ with $w(t) \in W$

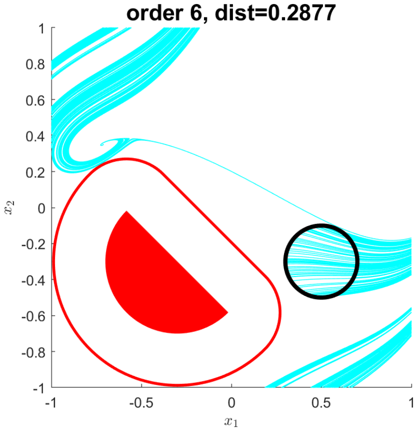
Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_f v(t, x, w), \mu \rangle$



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

Hybrid Systems

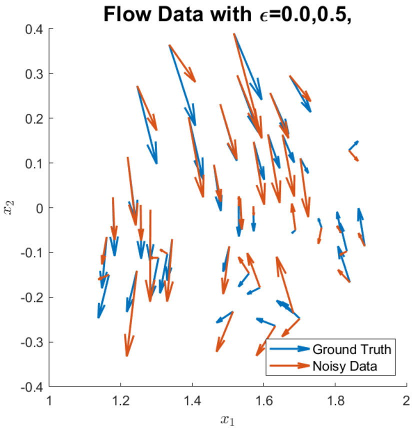
Continuous dynamics with discrete jumps/transitions



$$R_{\text{left}\to\text{bottom}} = [1 - x_2; x_1], \qquad R_{\text{right}\to\text{top}} = [x_2; x_1]$$

Sampling: Flow System

Data $\mathcal{D} = \{(t_j, x_j, \dot{x}_j)\}_j$ under mixed L_∞-bounded noise



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Dynamics Model

Given data $\mathcal{D}$, budget ϵ, system model $\{f_0, f_\ell\}$

Parameterize ground truth F by functions in dictionary

$$\dot{x}(t) = f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x)$$

Ground truth satisfies corruption $J(w^*) \leq \epsilon$

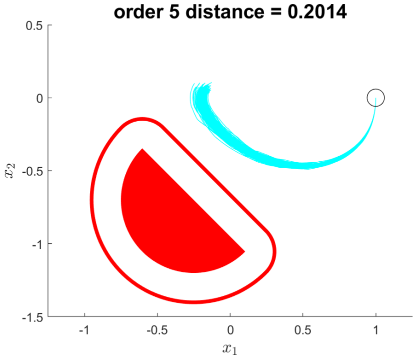
L_∞ example: $J(w) = \max_j \|f(t_j, x_j, w) - \dot{x}_j\|_\infty$

Distance Estimation Example (Flow)

Input-affine + Semidefinite Representable uncertainty

$$\mathcal{L}_f v(t, x, w) \leq 0 \qquad \forall (t, x, w) \in [0, T] \times X \times W$$

PSD Size 8568 $\to$ 56 ($L = 10$) using robust counterparts



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

Crash-Safety

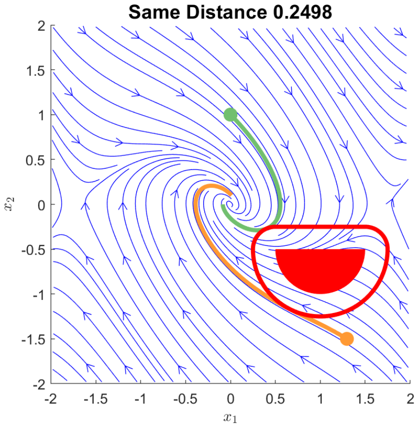
How much data corruption is needed to crash?

$$
\begin{aligned}
Q^* = \inf_{t^*,\ x_0,\ w} & \left[\sup_{t \in [0,t^*]} J(w(t))\right] \\
& \dot{x}(t) = f(t, x(t), w(t)) & \forall t \in [0, t^*] \\
& x(t \mid x_0, w(\cdot)) \in X_u \\
& w(\cdot) \in W,\ \ t^* \in [0, T],\ \ x_0 \in X_0
\end{aligned}
$$

Model safe if $Q^* > \epsilon$

Example Crash-Bounds

Two trajectories have same distance, different crash-bounds



Green-Top $Q^* = 0.316$, Yellow-Bottom $Q^* = 0.622$

Peak-Minimizing Control

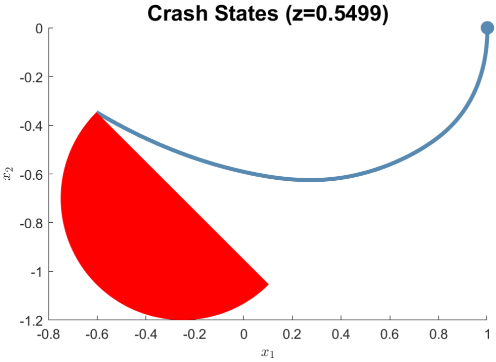
Add state $\dot{z} = 0$ (Molina, Rapaport, Ramírez 2022)

$$
\begin{aligned}
Q_z^* = \inf_{t^*,\ x_0,\ z,\ w} \quad & z && && \text{(4a)}\\
& \dot{x}(t) = f(t, x(t), w(t)) && \forall t \in [0, t^*] && \text{(4b)}\\
& \dot{z}(t) = 0 && \forall t \in [0, t^*] && \text{(4c)}\\
& J(w(t)) \leq z && \forall t \in [0, t^*] && \text{(4d)}\\
& x(t^* \mid x_0, w(\cdot)) \in X_u && && \text{(4e)}\\
& w(\cdot) \in W,\ \ t^* \in [0, T] && && \text{(4f)}\\
& x_0 \in X_0, z \in [0, J_{\max}] && && \text{(4g)}
\end{aligned}
$$

Equivalent formulation, $Q^* = Q_z^*$

Data-Driven Flow Crash-Bound

CasADi matches degree-4 moment-SOS crash bound



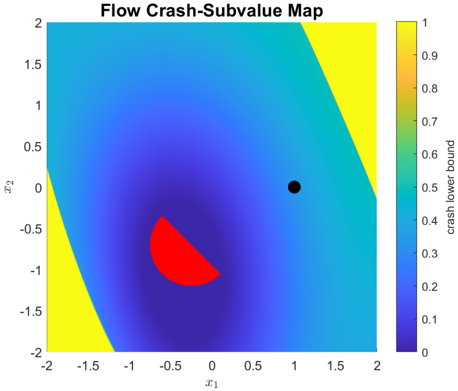
Terminal measure $\mu_p \in \mathcal{M}_+([0, T] \times X_u)$

True $\epsilon = 0.5 < 0.5499$, distance ≈ 0.2014

Flow Crash-Subvalue

Piecewise-polynomial subvalue for crash-safety

Based on Joint+Marginal optimization (Lasserre, 2010)



Bound of $0.3399 \leq 0.5499$, but valid everywhere in X

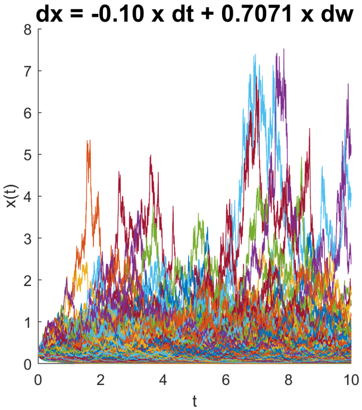
Peak Value-at-Risk Estimation

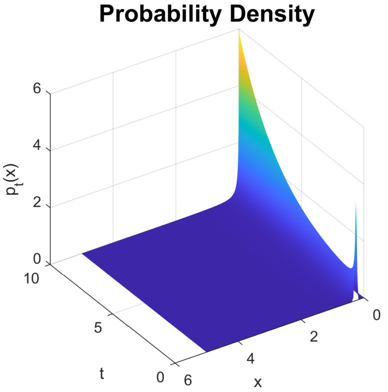
with M. Tacchi, M. Sznaier, A. Jasour

Stochastic Differential Equation

Multivariate SDE $dx = f(t,x)dt + g(t,x)dw$ (Itô)

Drift f and Diffusion g



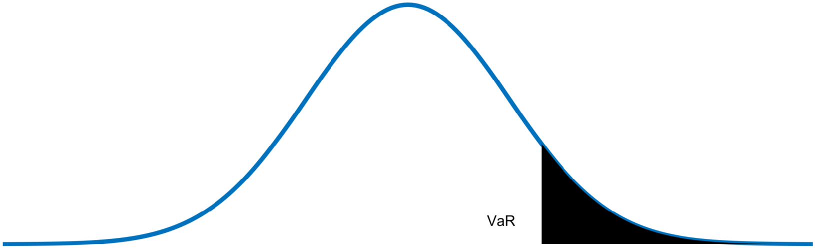


Geometric Brownian Motion

Value-at-Risk (Quantile)

ϵ-VaR of univariate measure $\omega(q)$ is unique number with

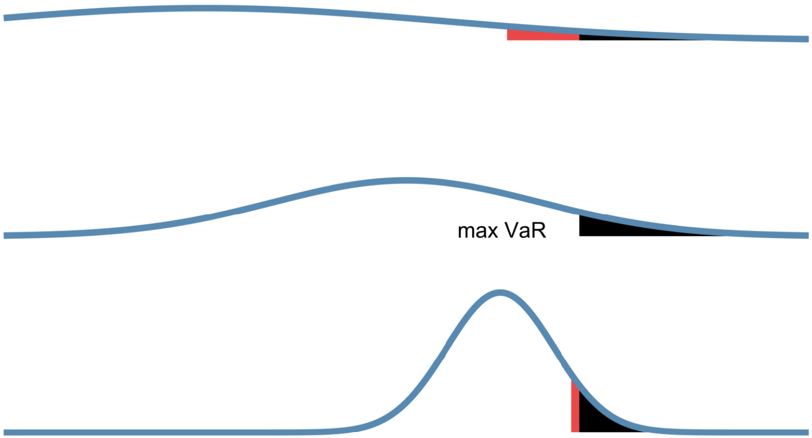
$$\text{Prob}_\omega(q \geq VaR_\epsilon(\omega)) = \epsilon$$



$VaR = 1.282$ for unit normal distribution at $\epsilon = 10\%$

Maximal Value at Risk

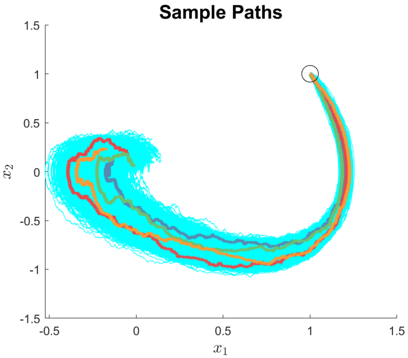
Maximize ϵ-VaR among multiple distributions

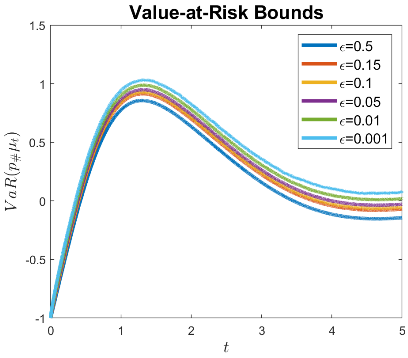


Red + Black areas = 10% probability

Value-at-Risk Example (Monte Carlo)

50,000 samples with $T = 5, \ \Delta t = 10^{-3}$





VaR of $p = -x_2$ along $dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$

Chance-Peak Problem

Maximize VaR of $p(x)$ along SDE trajectories

$p_{\#}\mu_{t^*}$: distribution of $p(x(t))$ at time t^*

$$
\begin{aligned}
P^* = \sup_{t^* \in [0,T]} \quad & VaR_{\epsilon}(p_{\#}\mu_{t^*}) && \text{(5a)}\\
& dx = f(t,x)dt + g(t,x)dw && \text{(5b)}\\
& \quad \text{stopping time of } \min(t^*, \text{ exit from } X) && \text{(5c)}\\
& x(0) \sim \mu_0. && \text{(5d)}
\end{aligned}
$$

Value-at-Risk Bounds

Concentration inequalities can upper-bound VaR

$$VaR_{\epsilon}(\omega) \leq \mathrm{stdev}(\omega) r + \mathrm{mean}(\omega)$$

Name	r	Condition
Cantelli	$\sqrt{1/(\epsilon) - 1}$	ω probability distribution
VP	$\sqrt{4/(9\epsilon) - 1}$	ω unimodal, $\epsilon < 1/6$

Coherent Risk Measures (e.g., CVaR) can also bound VaR

Concentration-Bounded Chance-Peak

Apply concentration inequalities to get upper bound $P_r^* \geq P^*$

Objective upper-bounds VaR w.r.t. time-t^* distribution μ_{t^*}

$$P_r^* = \sup_{t^* \in [0,T]} r\sqrt{\langle p^2, \mu_{t^*}\rangle - \langle p, \mu_{t^*}\rangle^2} + \langle p, \mu_{t^*}\rangle \tag{6a}$$

$$dx = f(t,x)dt + g(t,x)dw \tag{6b}$$

$$\text{stopping time of } \min(t^*, \text{exit from } X) \tag{6c}$$

$$x(0) \sim \mu_0. \tag{6d}$$

Max-Mean: $\epsilon = 0.5$, $r = 0$ (Cho, Stockbridge, 2002)

Occupation Measure Formulation

Occupation measure μ, terminal measure μ_τ

Second-Order Cone Program in measures (3d SOC)

$$p_r^* = \sup r\sqrt{\langle p^2, \mu_\tau\rangle - \langle p, \mu_\tau\rangle^2} + \langle p, \mu_\tau\rangle \tag{7a}$$

$$\mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu \tag{7b}$$

$$\mu_\tau,\ \mu \in \mathcal{M}_+([0, T] \times X) \tag{7c}$$

Generator $\mathcal{L}v = \partial_t v + f \cdot \nabla_x v + g^T(\nabla^2_{xx} v)g/2$ (Dynkin's)

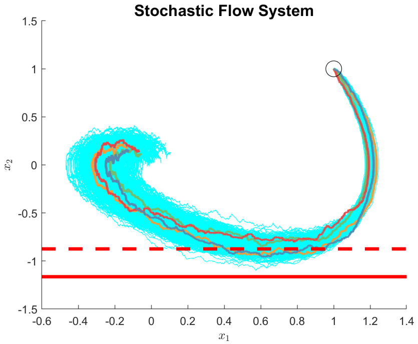
Results in upper-bound $p_r^* \geq P_r^* \geq P^*$, use moments

Chance-Peak Examples

Two-State

Stochastic Flow system from Prajna, Rantzer with $T = 5$

$$dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw.$$

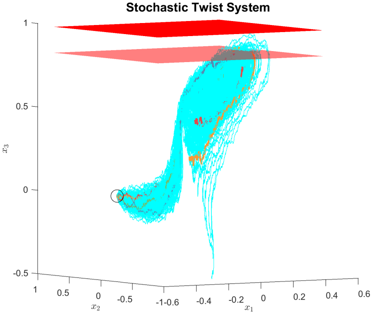


Maximize $-x_2$ with $d = 6$ (dashed=50%, solid=85% [ours])

Three-State

Stochasic Twist system with $T = 5$

$$dx = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} dw.$$

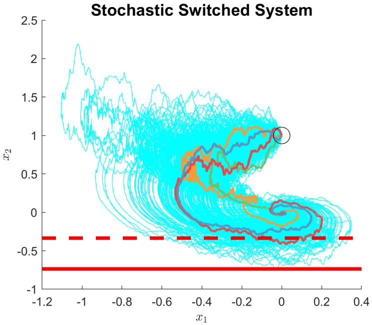


Maximize x_3 with $d = 6$ (translucent=50%, solid=85%)

Two-State Switching

Switching subsystems at $T = 5$

$$dx = \left\{ \begin{bmatrix} -2.5x_1 - 2x_2 \\ -0.5x_1 - x_2 \end{bmatrix}, \begin{bmatrix} -x_1 - 2x_2 \\ 2.5x_1 - x_2 \end{bmatrix} \right\} dt + \begin{bmatrix} 0 \\ 0.25x_2 \end{bmatrix} dw$$

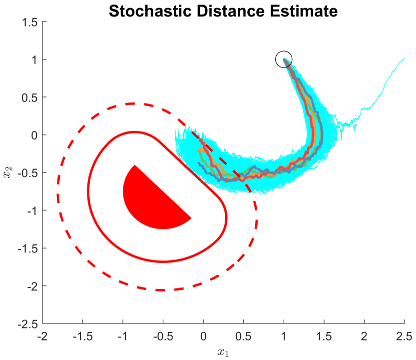


Maximize $-x_2$ with $d = 6$ (dashed=50%, solid=85%)

Two-State Distance

Half-circle unsafe set X_u

Based on distance estimation program



Minimize L_2 distance to X_u with $d = 6$ (dashed=50%, solid=85%)

Time-Delay Peak Estimation

with M. Korda, V. Magron, M. Sznaier

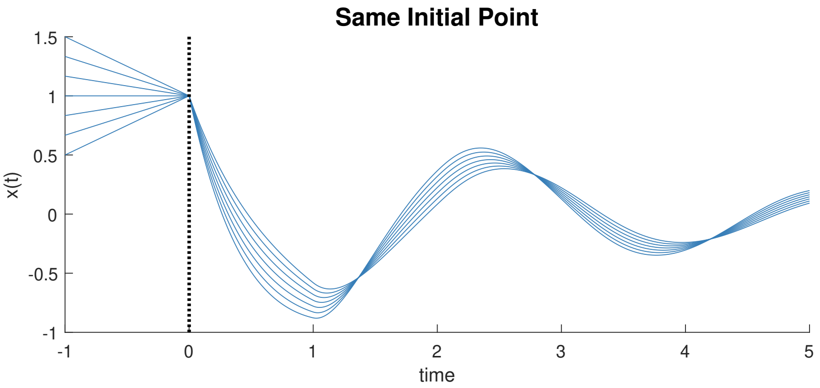
Time-Delay Examples

Delay between state change and its effect on system

$$\dot{x}(t) = f(t, x(t), x(t-\tau)) \qquad \forall t \in [0, T]$$
$$x(s) = x_h(s) \qquad \forall s \in [-\tau, 0]$$

System	**Delay**
Epidemic	Incubation Period
Population	Gestation Time
Traffic	Reaction Time
Congestion	Queue Time
Fluid Flow	Moving in Pipe

Dependence on History

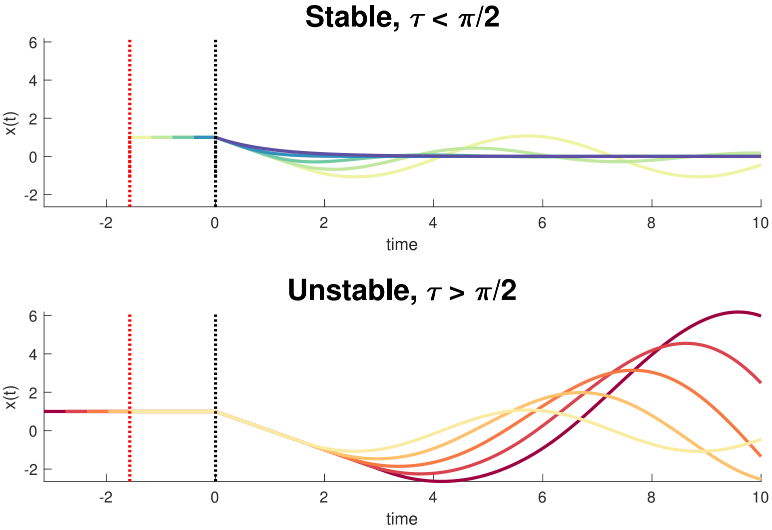


$$x'(t) = -2x(t) - 2x(t-1)$$

All trajectories pass through $(t, x) = (0, 1)$

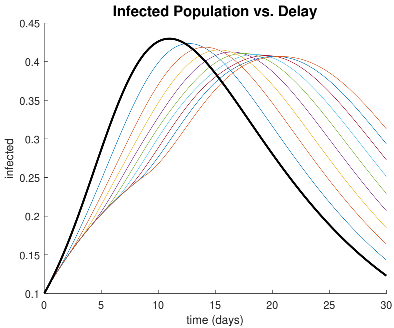
Initial history determines behavior, not just initial point

Delay Bifurcation Example

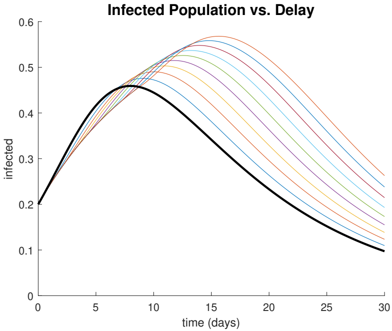


$$x'(t) = -x(t - \tau) \qquad \text{(Fridman 2014)}$$

Peak Value vs. Delay



(a) $I_h = 0.1$, peak decreases



(b) $I_h = 0.2$, peak increases

$$\begin{bmatrix} S'(t) \\ I'(t) \end{bmatrix} = \begin{bmatrix} -0.4S(t)I(t) \\ 0.4S(t-\tau)I(t-\tau) - 0.1I(t) \end{bmatrix}$$

Peak Estimation of Time-Delay Systems

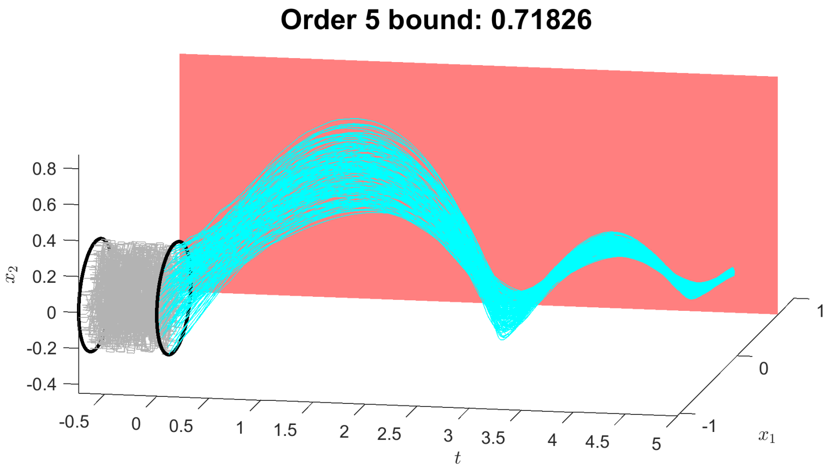
History $x_h(t)$ resides in a class of functions $\mathcal{H}$

Graph-constrained $\mathcal{H}$: $(t, x_h(t))$ contained in $H_0 \subset [-\tau, 0] \times X$

$$
\begin{aligned}
P^* = \sup_{t^*,\, x_h} \; & p(x(t^*)) \\
& \dot{x} = f(t, x(t), x(t-\tau)) && t \in [0, t^*] \\
& x(t) = x_h(t) && t \in [-\tau, 0] \\
& x_h(\cdot) \in \mathcal{H}
\end{aligned}
$$

Represent $x(t \mid x_h) : t \in [-\tau, t^*]$ as occupation measure

Time-Varying Preview



Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Existing Methods (very brief)

Certificates of Stability

- Lyapunov-Krasovskii
- Razumikhin
- LMI, Wirtinger
- ODE-Transport PDE

Relaxed control (Warga 1974, Vinter and Rosenblueth 1991-2)

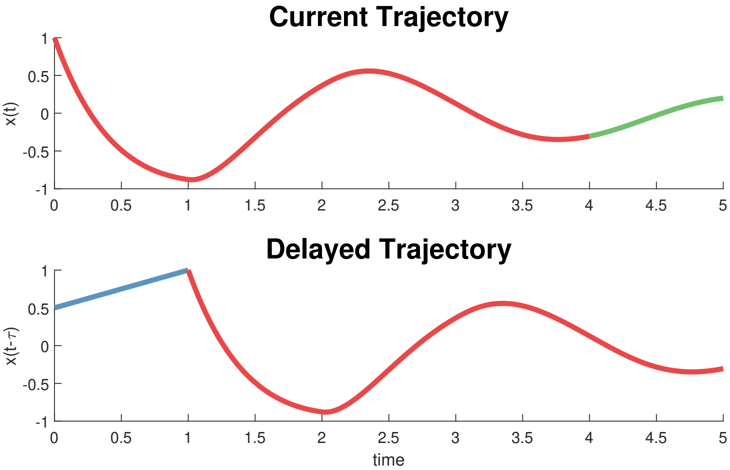
Fixed-terminal-time OCP with gridding (Barati 2012)

SOS Barrier (Papachristodoulou and Peet, 2010)

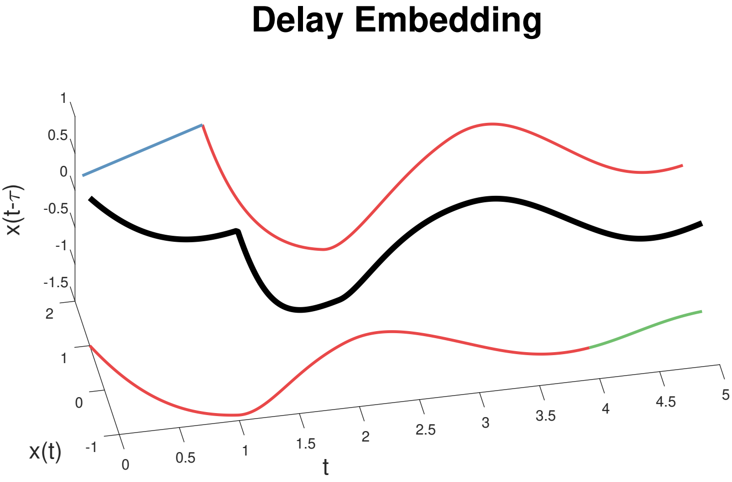
Riesz Operators (Magron and Prieur, 2020)

Time-Delay Measure Program

Time-Delay Visualization



$$x(t) = -2x(t) - 2x(t-1), \qquad x_h(t) = 1 - t/2$$



Black curve: $(t, x(t), x(t-\tau))$

Measure-Valued Solution

Tuple of measures for the delayed case

Peak	$\mu_p \in \mathcal{M}_+([0, T] \times X)$
Initial	$\mu_0 \in \mathcal{M}_+(X_0)$
History	$\mu_h \in \mathcal{M}_+(H_0)$
Occupation Start	$\bar{\mu}_0 \in \mathcal{M}_+([0, T - \tau] \times X^2)$
Occupation End	$\bar{\mu}_1 \in \mathcal{M}_+([T - \tau, T] \times X^2)$
Time-Slack	$\nu \in \mathcal{M}_+([0, T] \times X)$

Types of Constraints

Initial Conditions

Liouville: Dynamics

Consistency: Time-delay overlaps

Initial Conditions

Point evaluation $\langle 1, \mu_0 \rangle = 1$ at time $t = 0^+$

History $(t, x_h(t))$ defines a curve $[-\tau, 0]$, point at $x_h(0)$

t-marginal of μ_h should be the Lebesgue measure in $[-\tau, 0]$

Liouville

Treat $x(t-\tau) = x_1$ as an external input $\dot{x}_0 = f(t, x_0, x_1)$

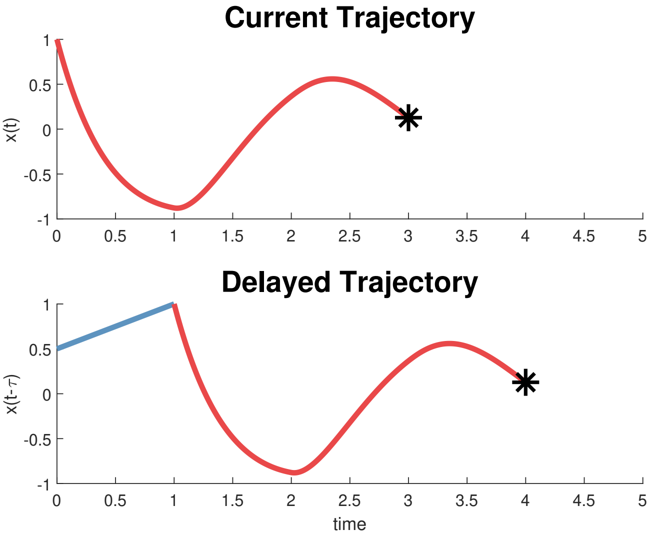
Sum $\bar{\mu} = \bar{\mu}_0 + \bar{\mu}_1$ in times $[0, T-\tau] \cap [T-\tau, T] = [0, T]$

Based on the delay embedding $(t, x(t), x(t-\tau))$

For all test functions $v \in C^1([0, T] \times X)$:

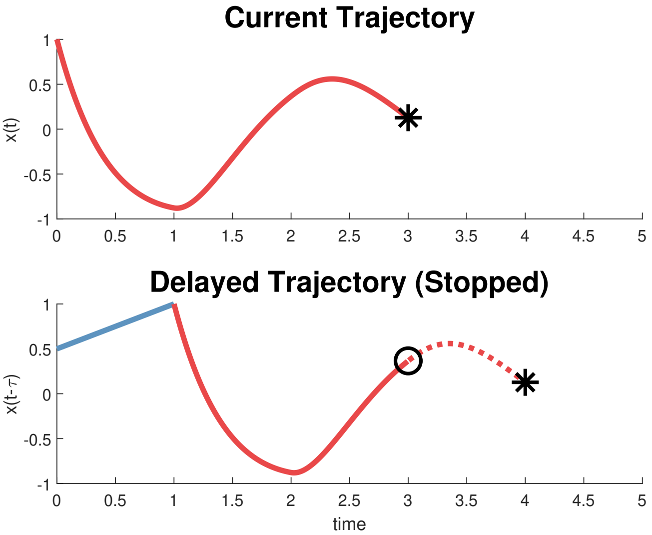
$$\langle v, \mu_p \rangle = \langle v(0, x), \mu_0(x) \rangle + \langle \mathcal{L}_{f(t, x_0, x_1)} v(t, x_0), \bar{\mu}(t, x_0, x_1) \rangle$$

Consistency Issue



Inconsistent elapsed times

Consistency Fix



Early stopping in delayed time

Consistency Constraint

Inspired by changing limits of integrals $t' \leftarrow t - \tau$

$$\left(\int_0^{t^*} + \int_{t^*}^{\min(T, t^*+\tau)}\right) \phi(t, x(t-\tau))dt$$
$$= \left(\int_{-\tau}^{0} + \int_0^{\min(t^*, T-\tau)}\right) \phi(t'+\tau, x(t'))dt'.$$

Shift-push $S^\tau_\#$ with $\langle \phi, S^\tau_\# \mu\rangle = \langle S^\tau \phi, \mu\rangle = \langle \phi(t+\tau, x), \mu\rangle$

Consistency constraint with time-slack ν

$$\pi^{tx_1}_\#(\bar{\mu}_0 + \bar{\mu}_1) + \nu = S^\tau_\#(\mu_h + \pi^{tx_0}_\# \bar{\mu}_0).$$

Measure Linear Program

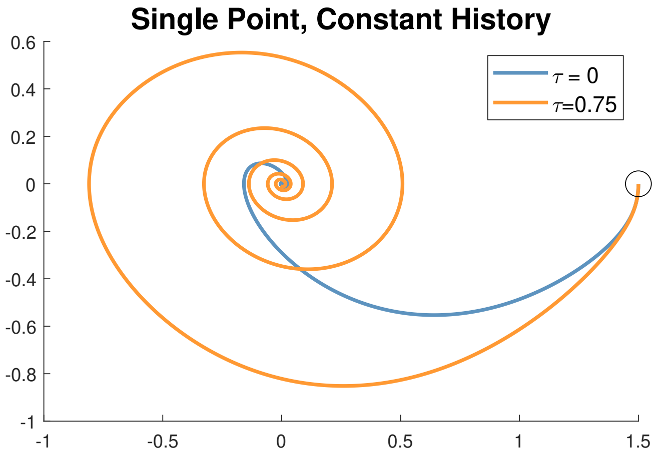
Linear program for time-delay peak estimation

$$
\begin{aligned}
p^* = \ \sup \quad & \langle p, \mu_p \rangle && \text{(8a)}\\
& \text{History-Validity}(\mu_0, \mu_h) && \text{(8b)}\\
& \text{Liouville}(\mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1) && \text{(8c)}\\
& \text{Consistency}(\mu_h, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(8d)}\\
& \text{Measure Definitions for } (\mu_h, \mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(8e)}
\end{aligned}
$$

Largest measures $\bar{\mu}_0, \bar{\mu}_1$ have $2n+1$ variables

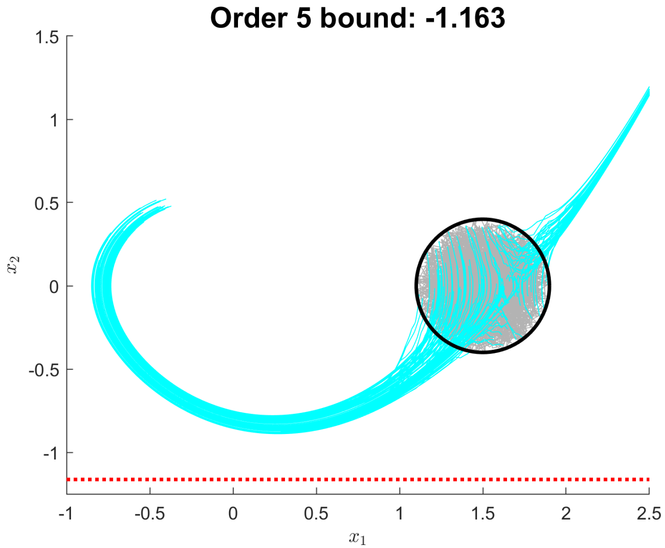
Time-Delay Examples

Delay Comparision



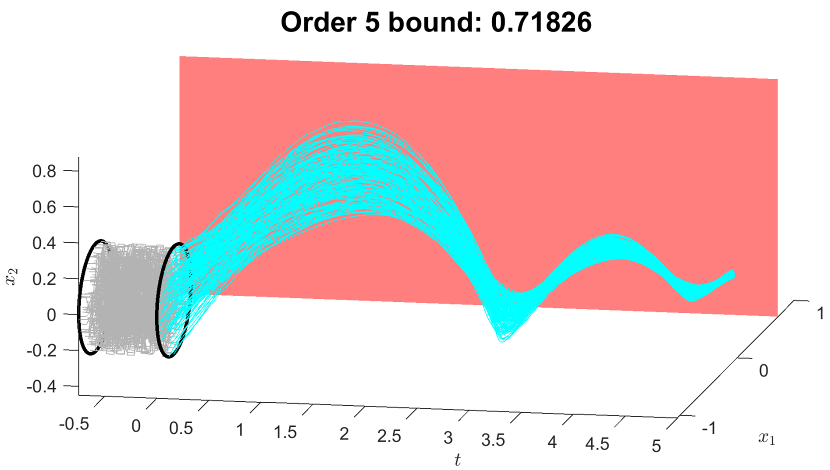
$$\dot{x}(t) = \begin{bmatrix} x_2 \\ -x_1(t-\tau) - x_2(t) + x_1(t)^3/3 \end{bmatrix}$$

Delayed Flow System



Minimize x_2 on the delayed Flow system

Time-Varying System (Reprise)



Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Take-aways

Summary

Noted importance of safety quantification

Extended occupation measure methods for peak estimation

Performed data-driven analysis using robust counterparts

Adapted to non-ODE systems (Hybrid, SDE, Time-Delay)

Future Work

- No-relaxation-gap for chance-peak and time-delay system
- High-order concentration inequalities
- Other time-delay models
- Lévy processes, Poisson jumps
- Distance-maximizing control
- Increased scalability, robotic systems
- Real-time computation

Safety is Important

Quantify using Peak Estimation

Journal Papers

Published:

1. J. Miller, D. Henrion, and M. Sznaier, "Peak Estimation Recovery and Safety Analysis," *IEEE Control Systems Letters*, vol. 5, no. 6, pp. 1982–1987, 2021 [link]

Conditionally Accepted:

1. J. Miller and M. Sznaier, "Bounding the Distance to Unsafe Sets with Convex Optimization," (Conditionally accepted by IEEE Transactions on Automatic Control in 2022) [link]

Conference Proceedings

1. J. Miller and M. Sznaier, "Bounding the Distance of Closest Approach to Unsafe Sets with Occupation Measures," in *2022 61st IEEE Conference on Decision and Control (CDC)*, pp. 5008–5013, 2022. [link]
2. J. Miller and M. Sznaier, "Facial Input Decompositions for Robust Peak Estimation under Polyhedral Uncertainty," *IFACPapersOnLine*, vol. 55, no. 25, pp. 55–60, 2022. [link]. IFAC Young Author Award (ROCOND)
3. J. Miller, D. Henrion, M. Sznaier, and M. Korda, "Peak Estimation for Uncertain and Switched Systems," in *2021 60th IEEE Conference on Decision and Control (CDC)*, pp. 3222–3228, 2021. [link]. Outstanding Student Paper Award (CDC 2021)

Preprints

1. J. Miller, M. Korda, V. Magron, and M. Sznaier "Peak Estimation of Time Delay Systems using Occupation Measures, " 2023. [link]
2. J. Miller, M. Tacchi, M. Sznaier, and A. Jasour, "Peak Value-at-Risk Estimation for Stochastic Differential Equations using Occupation Measures," 2023. [link]
3. J. Miller and M. Sznaier, "Peak Estimation of Hybrid Systems with Convex Optimization, " 2023. [link]
4. J. Miller and M. Sznaier "Quantifying the Safety of Trajectories using Peak-Minimizing Control, " 2023. [link]
5. J. Miller and M. Sznaier, "Analysis and Control of Input-Affine Dynamical Systems using Infinite-Dimensional Robust Counterparts," 2023. [link]

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Last but not least

The Warden

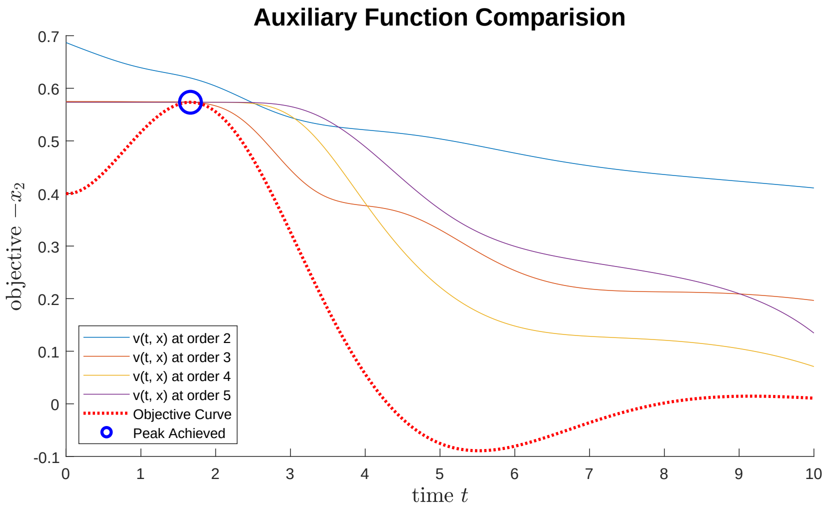
Thank you again for your attention

Thank you again for your attention

Cookies in Dana 429 (RSL)

Bonus: Data-Driven Program

Auxiliary Evaluation along Optimal Trajectory



Optimal $v(t,x)$ should be constant until peak is achieved

Noise Constraints

Polytopic region for L_∞-bounded noise

2 linear constraints for each coordinate i, sample j

$$-\epsilon \leq f_0(t_j, x_j)_i + \sum_{\ell=1}^{L} w_\ell f_\ell(t_j, x_j)_i - (\dot{x}_j)_i \leq \epsilon$$

Intersection of ellipsoids for L_2-bounded noise

$$\|f_0(t_j, x_j) + \sum_{\ell=1}^{L} w_\ell f_\ell(t_j, x_j) - (\dot{x}_j)\|_2 \leq \epsilon$$

Robust Counterpart Theory

Semidefinite-representable uncertainty set

$$W = \cap_s \{\exists \lambda_s \in \mathbb{R}^{q_s} : A_s w + G_s \lambda_s + e_s \in K_s\}$$

Lie constraint (based on Ben-Tal, Nemirovskii, 2009)

$$\mathcal{L}_f v(t, x, w) \leq 0 \qquad \forall (t, x, w) \in [0, T] \times X \times W.$$

Nonconservative robust counterpart with multipliers ζ

$$\begin{aligned}
&\mathcal{L}_{f_0} v(t, x) + \textstyle\sum_{s=1}^{N_s} e_s^T \zeta_s(t, x) \leq 0 && \forall [0, T] \times X \\
&G_s^T \zeta_s(t, x) = 0 && \forall s = 1..N_s \\
&\textstyle\sum_{s=1}^{N_s} (A_s^T \zeta_s(t, x))_\ell + f_\ell(t, x) \cdot \nabla_x v(t, x) = 0 && \forall \ell = 1..L \\
&\zeta_s(t, x) \in K_s^* && \forall s = 1..N_s
\end{aligned}$$

Peak Decomposed Program

Example: Polytopic uncertainty $W = \{w \mid Aw \leq b\}$

Only the Lie Derivative constraint changes

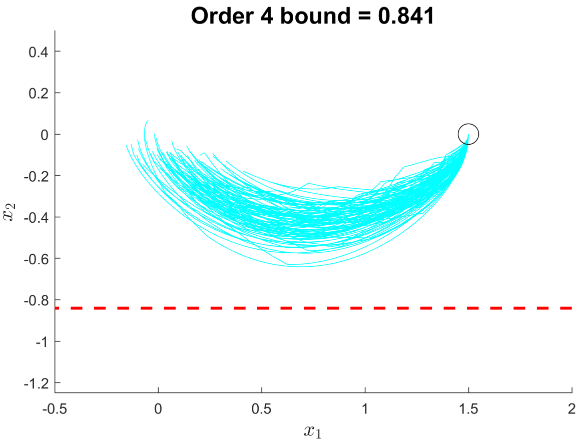
$$
\begin{aligned}
d^* = \ & \min_{\gamma \in \mathbb{R}} \gamma \\
& \gamma \geq v(0, x) & & \forall x \in X_0 \\
& \mathcal{L}_{f_0} v(t, x) + b^T \zeta(t, x) \leq 0 & & \forall (t, x) \in [0, T] \times X \\
& (A^T)_\ell \zeta(t, x) = (f_\ell \cdot \nabla_x) v(t, x) & & \forall \ell = 1..L \\
& v(t, x) \geq p(x) & & \forall (t, x) \in [0, T] \times X \\
& v(t, x) \in C^1([0, T] \times X) & & \\
& \zeta_k(t, x) \in C_+([0, T] \times X) & & \forall k = 1..m
\end{aligned}
$$

Peak Estimation Example (Flow)

$$\dot{x} = [x_2,\ -wx_1 - x_2 + x_1^3/3]$$

$$L = 1,\ m = 80 \text{ (2 nonredundant)}$$

Peak Estimation Example (Flow)



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

$$L = 10,\ \ m = 80\ (33\ \text{nonredundant})$$

Crash-Bound Program

Consistency sets

$$Z = [0, J_{\max}] \qquad \Omega = \{(w, z) \in W \times Z : J(w) \leq z\}.$$

Optimal Control Problem with auxiliary $v(t, x, z) \in C^1$

$$\begin{aligned}
d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\
& v(0, x, z) \geq \gamma && \forall (x, z) \in X_0 \times Z \\
& v(t, x, z) \leq z && \forall (t, x, z) \in [0, T] \times X_u \times Z \\
& \mathcal{L}_f v(t, x, z, w) \geq 0 && \forall (t, x, z, w) \in [0, T] \times X \times \Omega
\end{aligned}$$

Crash Lie-decomposition

Exploit affine structure of $J(w) = \|\Gamma w - h\|_\infty$

Nonconservatively robustified Lie constraint

$$
\begin{aligned}
d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\
& v(0, x, z) \geq \gamma && \forall (x, z) \in X_0 \times Z \\
& v(t, x, z) \leq z && \forall (t, x, z) \in [0, T] \times X_u \times Z \\
& \mathcal{L}_{f_0} v - (z\mathbf{1} + h)^T \zeta \geq 0 && \forall (t, x, z) \in [0, T] \times X \times [0, J_{\max}] \\
& (\Gamma^T)_\ell \zeta + f_\ell \cdot \nabla_x v = 0 && \forall \ell = 1..L \\
& \zeta_j \in C_+([0, T] \times X \times Z) && \forall j = 1..2nT.
\end{aligned}
$$

Sum-of-Squares Method

Every $c \in \mathbb{R}$ satisfies $c^2 \geq 0$

Sufficient: $q(x) \in \mathbb{R}[x]$ nonnegative if $q(x) = \sum_i q_i^2(x)$

Exists $v(x) \in \mathbb{R}[x]^s$, *Gram* matrix $Z \in \mathbb{S}_+^s$ with $q = v^T Z v$

Sum-of-Squares (SOS) cone $\Sigma[x]$

$$\begin{aligned} &x^2y^4 - 6x^2y^2 + 10x^2 + 2xy^2 + 4xy - 6x + 4y^2 + 1 \\ =&(x + 2y)^2 + (3x - 1 - xy^2)^2 \end{aligned}$$

Motzkin Counterexample (nonnegative but not SOS)

$$x^2y^4 + x^4y^2 - x^2y^2 + 1$$

Sum-of-Squares Method (cont.)

Putinar Positivestellensatz (Psatz) nonnegativity certificate over set $\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}$:

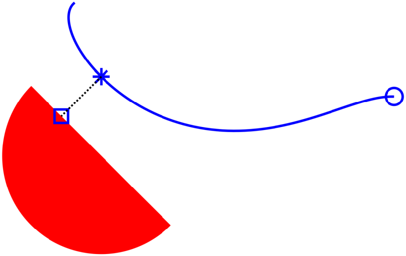
$$q(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x) \tag{9a}$$

$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma_i(x) \in \Sigma[x], \quad \phi_j \in \mathbb{R}[x]. \tag{9b}$$

Psatz at degree $2d$ is an SDP, monomial basis: $s = \binom{n+d}{d}$

Archimedean: $\exists R \geq 0$ where $R - \|x\|_2^2$ has Psatz over $\mathbb{K}$

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*,\ x_0^*,\ t_p^*)$:

x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

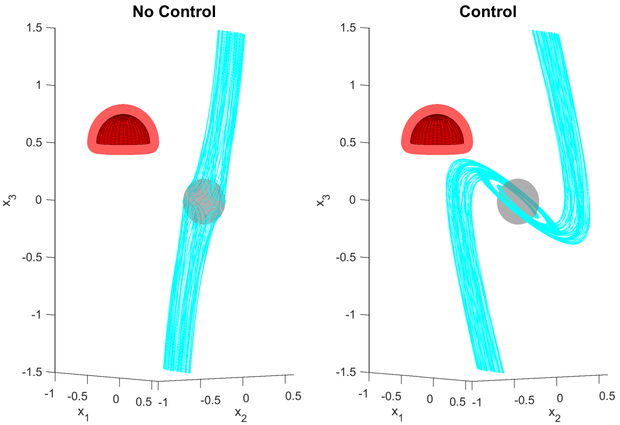
$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0,T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \textstyle\int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Hybrid Systems

State guards and transitions



L_2 bound 0.0891: uncontrolled to boundary, controlled to sphere

Bonus: Chance-Peak

Second-Order Cone Program

Reformulate as infinite-dimensional second-order cone program

SOC set $Q^3 = \{(s, \kappa) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \mid \|s\|_2 \leq \kappa\}$

$$
\begin{aligned}
p_r^* = \sup_{z \in \mathbb{R}} \quad & rz + \langle p, \mu_\tau \rangle && \text{(10a)} \\
& \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu && \text{(10b)} \\
& s = [1 - \langle p^2, \mu_\tau \rangle, \ 2z, \ 2\langle p, \mu_\tau \rangle] && \text{(10c)} \\
& (s, 1 + \langle p^2, \mu_\tau \rangle) \in Q^3 && \text{(10d)} \\
& \mu, \ \mu_\tau \in \mathcal{M}_+([0, T] \times X). && \text{(10e)}
\end{aligned}
$$

Moment-SOS: $p_d^* \geq p_{d+1}^* \geq \ldots \geq p_r^* = P_r^* \geq P^*$

Bonus: Time Delay

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

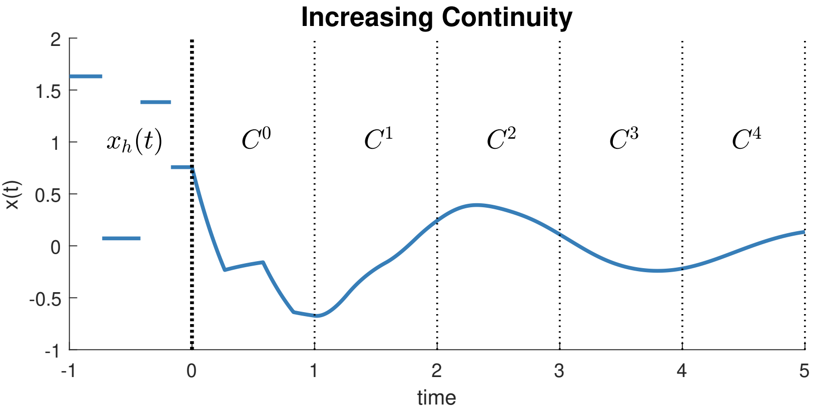
Degree d, dynamics degree $\tilde{d} = d + \max(\lfloor \deg f/2 \rfloor, \deg g - 1)$

Bounds: $p^*_d \geq p^*_{d+1} \geq \ldots \geq p^*_r = P^*_r \geq P^*$

Measure	$\mu_p(t,x)$	$\mu(t,x)$
PSD Size	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(1+n)^{6\tilde{d}}$ or $\tilde{d}^{4(n+1)}$

Propagation of Continuity



$$x'(t) = -2x(t) - 2x(t-1)$$

Continuity increases every τ_r time steps

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lfloor \deg f/2 \rfloor$

Bounds: $p^*_d \geq p^*_{d+1} \geq \ldots = p^* \geq P^*$

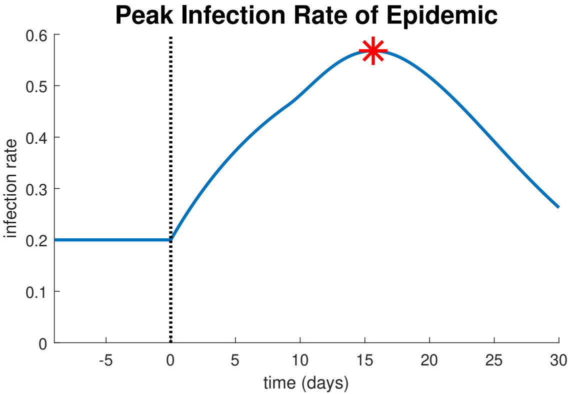
Size of Moment Matrices Peak Estimation

Measure:	μ_0	μ^p	μ_h
Size:	$\binom{n+d}{d}$	$\binom{n+1+d}{d}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Measure:	$\bar{\mu}_0$	$\bar{\mu}_1$	ν
Size:	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(2n+1)^{6\tilde{d}}$ or $\tilde{d}^{4(2n+1)}$

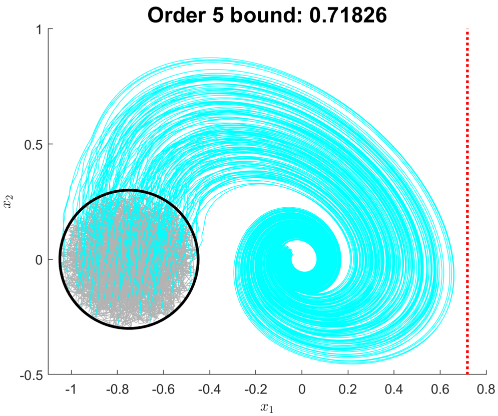
SIR Peak Estimation Example



Upper bound $I_{max} \geq 56.9\%$ with order 3 LMI

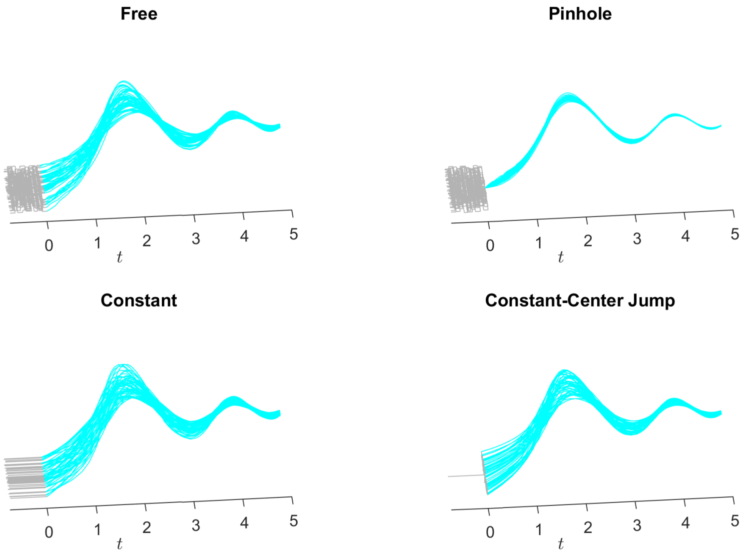
Recovery: $t_* = 15.6$ days, $(S^*, I^*) = (56.9\%, 5.61\%)$

Time-Varying System



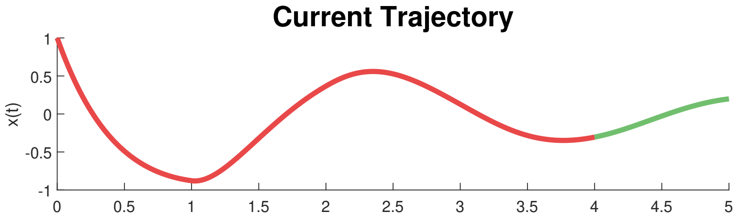
$$\text{Maximize } x_1 \text{ on } \dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$$

Time-Varying Histories



History restrictions and trajectories of system

Joint+Component Consistency

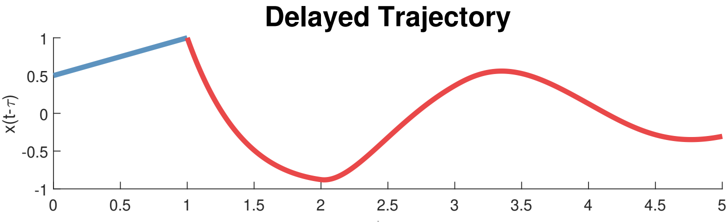


(t, x_0) marginal of $\bar{\mu}$

For all test functions $\phi_0 \in C([0, T] \times X)$

$$\begin{aligned} \langle \phi_0(t, x_0), \bar{\mu} \rangle &= \int_0^T \phi_0(t, x(t \mid x_h))dt \\ &= \left(\int_0^{T-\tau} + \int_{T-\tau}^T \right) \phi_0(t, x(t \mid x_h))dt \\ &= \langle \phi_0(t, x), \nu_0 + \nu_1 \rangle \end{aligned}$$

Joint+Component Consistency (cont.)



(t, x_1) marginal of $\bar{\mu}$

For all test functions $\phi_1 \in C([0, T] \times X)$

$$\begin{aligned}
\langle \phi_1(t, x_1), \bar{\mu} \rangle &= \int_0^T \phi_1(t, x(t - \tau \mid x_h))dt \\
&= \int_{-\tau}^{T-\tau} \phi_1(t + \tau, x(t \mid x_h))dt \\
&= \int_{-\tau}^{0} \phi_1(t + \tau, x_h(t))dt + \langle \phi_1(t + \tau, x), \nu_0 \rangle
\end{aligned}$$

Joint+Component Experiment

Table 1: Objective values for Flow experiment

degree d	1	2	3	4	5
Joint+Component	1.25	1.223	1.1937	1.1751	1.1636
Standard	1.25	1.2183	1.1913	1.1727	1.1630

Table 2: Time (seconds) to obtain SDP bounds in Table 1

degree d	1	2	3	4	5
Joint+Component	0.782	0.991	5.271	31.885	336.509
Standard	0.937	1.190	9.508	105.777	552.496

Bonus: Measure Background

Measures

Nonnegative Borel Measure μ

Assigns each set $A \subseteq X$ a 'size' $\mu(A) \geq 0$ (Measure)

Mass $\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

$\mu \in \mathcal{M}_+(X)$: space of measures on X

$f \in C(X)$: continuous function on X

Pairing by Lebesgue integration $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$

Dirac Delta Measure

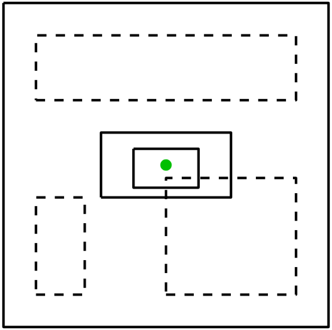
Dirac delta $\delta_{x'}(A) = \begin{cases} 1 & x' \in A \\ 0 & x' \notin A \end{cases}$

Probability:

$\delta_{x'}(X) = 1, \ \langle f(x), \delta_{x'} \rangle = f(x')$

$\mu(A) = 1$: Solid Box

$\mu(A) = 0$: Dashed Box



Atomic Measure

Rank-1 atomic measure

$$\mu = c\delta_{x'} \qquad c > 0$$

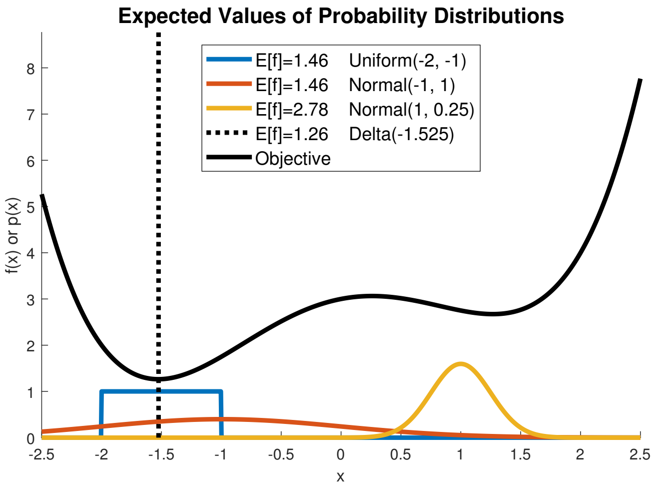
Rank-2 atomic measure

$$\mu = c_1\delta_{x_1'} + c_2\delta_{x_2'} \qquad c > 0,\ x_1' \neq x_2'$$

Rank-r atomic measure

$$\mu = \sum_{i=1}^{r} c_i\delta_{x_i'} \qquad c > 0,\ \{x_i'\}_{i=1}^{r}\text{distinct}$$

Example of Measure Optimization

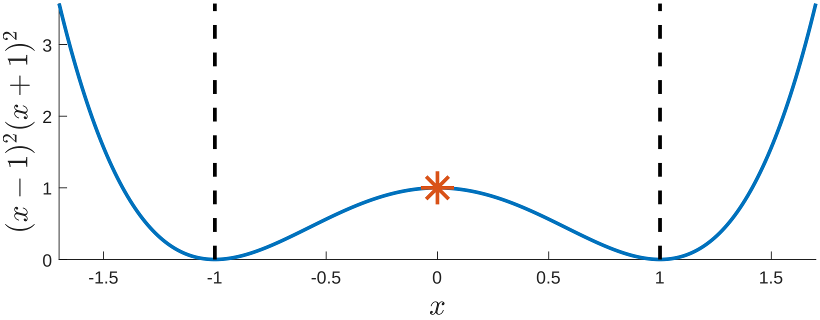


Optimum $\mathbb{E}_\mu[f] = \langle f, \mu \rangle$ at $\mu = \delta_{x^*}$

Measure Optimization

Nonconvex problems could be convex in measures

$$\min_{x \in K} p(x) \to \min_{\mu \in \mathcal{M}_+(K)} \langle p, \mu \rangle, \quad \langle 1, \mu \rangle = 1$$



$f(\frac{1}{2}(1 + (-1))) = 1$, but $\frac{1}{2}(f(1) + f(-1)) = 0$

Bonus: Approximating Measure LPs

Need for Approximation

Measure LPs are infinite-dimensional

Linear Matrix Inequality: convex problem

$$\max_y b^T y \qquad C + \sum_{i=1}^{m} A_i y_i \geq 0$$

Solve LMIs through (interior point, ADMM, etc.)

Approximate infinite LPs by finite-dimensional LMIs

Moments

Monomial $x^\alpha = \prod_i x_i^{\alpha_i}$ for power $\alpha \in \mathbb{N}^n$

Degree $|\alpha| = \sum_i \alpha_i$

α-moment of measure $y_\alpha = \langle y_\alpha, \mu \rangle$

Measure uniquely described by infinite set $\{y_\alpha\}_{\alpha \in \mathbb{N}^n}$

When does a sequence $\{y_\alpha\}_{\alpha \in \mathcal{A}}$ correspond to a measure μ?

Linear Functional

Linear Functional polynomial $\rightarrow$ moments

$$f(x) \rightarrow \int_X f(x) d\mu = \int_X \sum_\alpha f_\alpha x^\alpha d\mu = \sum_\alpha f_\alpha y_\alpha$$

Bivariate Example

$$2 + x_1 x_2 - 3x_1^2 + x_1 x_2^3 \rightarrow 2 + y_{11} - 3y_{20} + y_{13}$$

Moment Matrices

Squares $f(x)^2$ are nonnegative (real)

$f(x)^2 \geq 0$ implies that $\langle f(x)^2, \mu \rangle \geq 0 \quad \forall f \in \mathbb{R}[x]$:

$$\langle f(x)^2, \mu \rangle = \int_X \sum_{\alpha,\beta} (f_\alpha x^\alpha)(f_\beta x^\beta) d\mu = \int_X \sum_{\alpha,\beta} (f_\alpha f_\beta x^{\alpha+\beta}) d\mu \geq 0$$

Moment matrix $\mathbb{M}[y] \succeq 0$ has $\mathbb{M}[y]_{\alpha,\beta} = y_{\alpha+\beta}$

$$\langle f(x)^2, \mu \rangle = \mathbf{f}^T \mathbb{M}[y] \mathbf{f} \geq 0$$

Moment Matrix Example

Moments up to degree $2 \times 2 = 4$

$$\mathbb{M}_2[y] = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{11} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}$$

Localizing Matrices

μ supported on set $K = \{x \mid g_i(x) \geq 0, i = 1 \ldots N\}$

$g_i(x)f(x)^2 \geq 0$ implies that $\langle g_i(x)f(x)^2, \mu\rangle \geq 0$

$$\langle g_i(x)f(x)^2, \mu\rangle = \int_X \sum_{\alpha,\beta,\gamma} (f_\alpha f_\beta g_\gamma x^{\alpha+\beta+\gamma}) d\mu \geq 0$$

Localizing matrix $\mathbb{M}[g_i m] \succeq 0$ has $\mathbb{M}[g_i m]_{\alpha,\beta} = \sum_\gamma g_\gamma m_{\alpha+\beta+\gamma}$

$$\langle g_i(x)f(x)^2, \mu\rangle = \mathbf{f}^T \mathbb{M}[g_i y]\mathbf{f} \geq 0$$

Moment-SOS Hierarchy

Polynomial optimization problem example :

$$p^* = \max_{x \in K} p(x) = \max_{\mu \in \mathcal{M}_+(K)} \langle p(x), \mu \rangle, \quad \mu(K) = 1$$

Keep moments up to degree d:

$$p_d^* = \max_{y} \ \sum_{|\alpha| \leq 2d} p_\alpha m_\alpha$$

$$\mathbb{M}_d[y],\ \mathbb{M}_{d-\deg(g_i)}[g_i y] \succeq 0$$

Finite-dimensional SDP: $\mathbb{M}_d[y]$ has size $\binom{n+d}{d}$

Bounds $p_d^* \geq p_{d+1}^* \geq p_{d+2}^* \dots$ converge to p^* as $d \to \infty$

Approximation Pipeline

1. Trajectory Program
2. Measure LP
3. Moment LMI

Increase degree d of LMI to get better bounds

Prove conditions under which $\lim_{d \to \infty} p^*_d \to p^* = P^*$