Safety Quantification for Nonlinear and Time-Delay Systems using Occupation Measures (Bonus Content)

Author: **Jared Miller**

Committee: Octavia Camps
Didier Henrion (LAAS-CNRS)
Bahram Shafai
Eduardo Sontag
Mario Sznaier

Northeastern University

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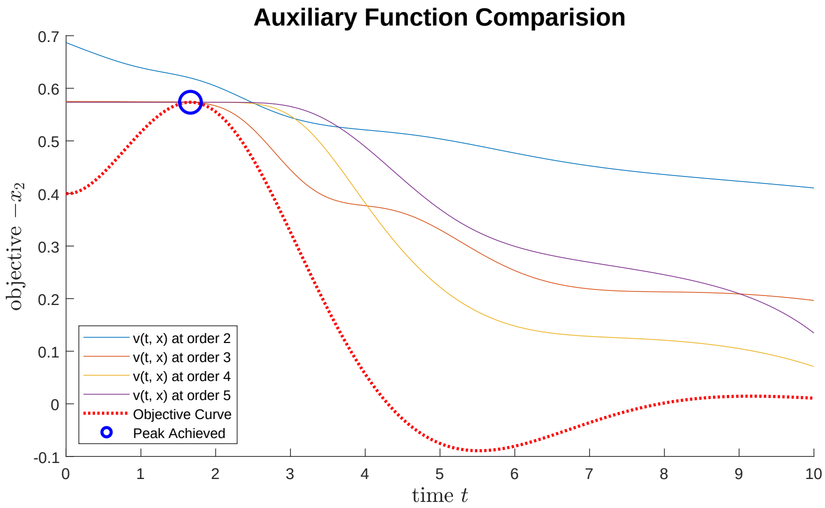
Bonus: Data-Driven Program

Auxiliary Evaluation along Optimal Trajectory



Optimal $v(t, x)$ should be constant until peak is achieved

Noise Constraints

Polytopic region for L_∞-bounded noise

2 linear constraints for each coordinate i, sample j

$$-\epsilon \leq f_0(t_j, x_j)_i + \sum_{\ell=1}^{L} w_\ell f_\ell(t_j, x_j)_i - (\dot{x}_j)_i \leq \epsilon$$

Intersection of ellipsoids for L_2-bounded noise

$$\| f_0(t_j, x_j) + \sum_{\ell=1}^{L} w_\ell f_\ell(t_j, x_j) - (\dot{x}_j) \|_2 \leq \epsilon$$

Robust Counterpart Theory

Semidefinite-representable uncertainty set

$$W = \cap_s \{\exists \lambda_s \in \mathbb{R}^{q_s} : A_s w + G_s \lambda_s + e_s \in K_s\}$$

Lie constraint (based on Ben-Tal, Nemirovskii, 2009)

$$\mathcal{L}_f v(t, x, w) \leq 0 \qquad \forall (t, x, w) \in [0, T] \times X \times W.$$

Nonconservative robust counterpart with multipliers ζ

$$\begin{aligned}
&\mathcal{L}_{f_0} v(t, x) + \textstyle\sum_{s=1}^{N_s} e_s^T \zeta_s(t, x) \leq 0 && \forall [0, T] \times X \\
&G_s^T \zeta_s(t, x) = 0 && \forall s = 1..N_s \\
&\textstyle\sum_{s=1}^{N_s} (A_s^T \zeta_s(t, x))_\ell + f_\ell(t, x) \cdot \nabla_x v(t, x) = 0 && \forall \ell = 1..L \\
&\zeta_s(t, x) \in K_s^* && \forall s = 1..N_s
\end{aligned}$$

Peak Decomposed Program

Example: Polytopic uncertainty $W = \{w \mid Aw \leq b\}$

Only the Lie Derivative constraint changes

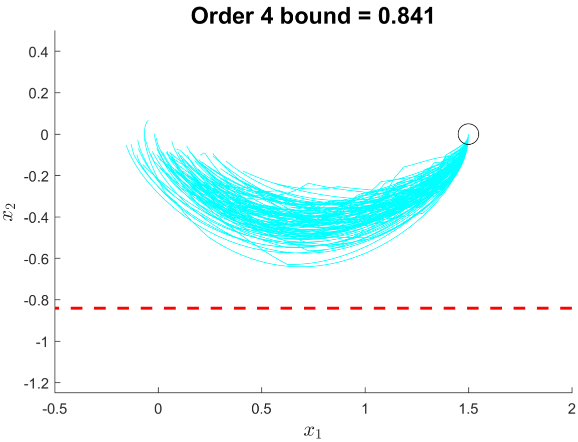
$$
\begin{aligned}
d^* = \ & \min_{\gamma \in \mathbb{R}} \gamma \\
& \gamma \geq v(0, x) & & \forall x \in X_0 \\
& \mathcal{L}_{f_0} v(t, x) + b^T \zeta(t, x) \leq 0 & & \forall (t, x) \in [0, T] \times X \\
& (A^T)_\ell \zeta(t, x) = (f_\ell \cdot \nabla_x) v(t, x) & & \forall \ell = 1..L \\
& v(t, x) \geq p(x) & & \forall (t, x) \in [0, T] \times X \\
& v(t, x) \in C^1([0, T] \times X) & & \\
& \zeta_k(t, x) \in C_+([0, T] \times X) & & \forall k = 1..m
\end{aligned}
$$

Peak Estimation Example (Flow)

$$\dot{x} = [x_2, \; -wx_1 - x_2 + x_1^3/3]$$

$$L = 1, \; m = 80 \text{ (2 nonredundant)}$$

Peak Estimation Example (Flow)



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

$$L = 10,\ \ m = 80\ (33\ \text{nonredundant})$$

Crash-Bound Program

Consistency sets

$$Z = [0, J_{\max}] \qquad \Omega = \{(w, z) \in W \times Z : J(w) \leq z\}.$$

Optimal Control Problem with auxiliary $v(t, x, z) \in C^1$

$$\begin{aligned} d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\ & v(0, x, z) \geq \gamma && \forall (x, z) \in X_0 \times Z \\ & v(t, x, z) \leq z && \forall (t, x, z) \in [0, T] \times X_u \times Z \\ & \mathcal{L}_f v(t, x, z, w) \geq 0 && \forall (t, x, z, w) \in [0, T] \times X \times \Omega \end{aligned}$$

Crash Lie-decomposition

Exploit affine structure of $J(w) = \|\Gamma w - h\|_\infty$

Nonconservatively robustified Lie constraint

$$
\begin{aligned}
d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\
& v(0, x, z) \geq \gamma && \forall (x, z) \in X_0 \times Z \\
& v(t, x, z) \leq z && \forall (t, x, z) \in [0, T] \times X_u \times Z \\
& \mathcal{L}_{f_0} v - (z\mathbf{1} + h)^T \zeta \geq 0 && \forall (t, x, z) \in [0, T] \times X \times [0, J_{\max}] \\
& (\Gamma^T)_\ell \zeta + f_\ell \cdot \nabla_x v = 0 && \forall \ell = 1..L \\
& \zeta_j \in C_+([0, T] \times X \times Z) && \forall j = 1..2nT.
\end{aligned}
$$

Sum-of-Squares Method

Every $c \in \mathbb{R}$ satisfies $c^2 \geq 0$

Sufficient: $q(x) \in \mathbb{R}[x]$ nonnegative if $q(x) = \sum_i q_i^2(x)$

Exists $v(x) \in \mathbb{R}[x]^s$, *Gram* matrix $Z \in \mathbb{S}_+^s$ with $q = v^T Z v$

Sum-of-Squares (SOS) cone $\Sigma[x]$

$$\begin{aligned} & x^2y^4 - 6x^2y^2 + 10x^2 + 2xy^2 + 4xy - 6x + 4y^2 + 1 \\ =& (x + 2y)^2 + (3x - 1 - xy^2)^2 \end{aligned}$$

Motzkin Counterexample (nonnegative but not SOS)

$$x^2y^4 + x^4y^2 - x^2y^2 + 1$$

Sum-of-Squares Method (cont.)

Putinar Positivestellensatz (Psatz) nonnegativity certificate over set $\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}$:

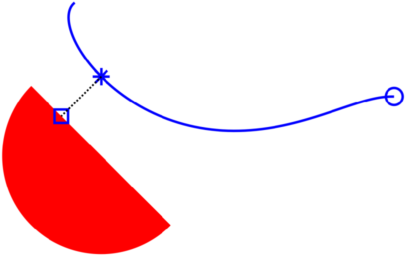
$$q(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x) \tag{1a}$$

$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma_i(x) \in \Sigma[x], \quad \phi_j \in \mathbb{R}[x]. \tag{1b}$$

Psatz at degree $2d$ is an SDP, monomial basis: $s = \binom{n+d}{d}$

Archimedean: $\exists R \geq 0$ where $R - \|x\|_2^2$ has Psatz over $\mathbb{K}$

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*,\ x_0^*,\ t_p^*)$:

x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

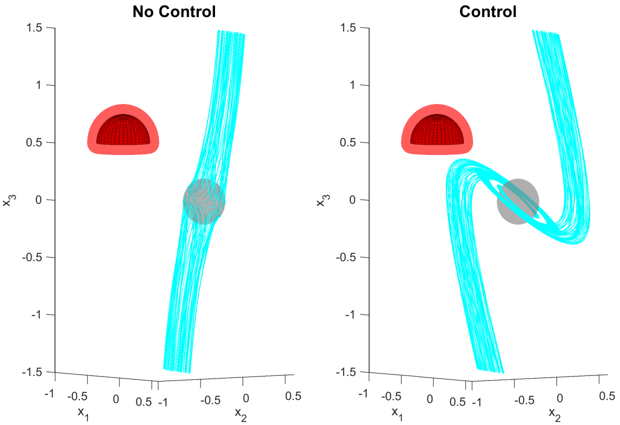
$$\mu_0^* : \qquad \delta_{x=x_0^*}$$
$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$
$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0, T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Hybrid Systems

State guards and transitions



L_2 bound 0.0891: uncontrolled to boundary, controlled to sphere

Bonus: Chance-Peak

Second-Order Cone Program

Reformulate as infinite-dimensional second-order cone program

SOC set $Q^3 = \{(s, \kappa) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \mid \|s\|_2 \leq \kappa\}$

$$p_r^* = \sup_{z \in \mathbb{R}} \quad rz + \langle p, \mu_\tau \rangle \tag{2a}$$

$$\mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu \tag{2b}$$

$$s = [1 - \langle p^2, \mu_\tau \rangle,\ 2z,\ 2\langle p, \mu_\tau \rangle] \tag{2c}$$

$$(s, 1 + \langle p^2, \mu_\tau \rangle) \in Q^3 \tag{2d}$$

$$\mu,\ \mu_\tau \in \mathcal{M}_+([0, T] \times X). \tag{2e}$$

Moment-SOS: $p_d^* \geq p_{d+1}^* \geq \ldots \geq p_r^* = P_r^* \geq P^*$

Bonus: Time Delay

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

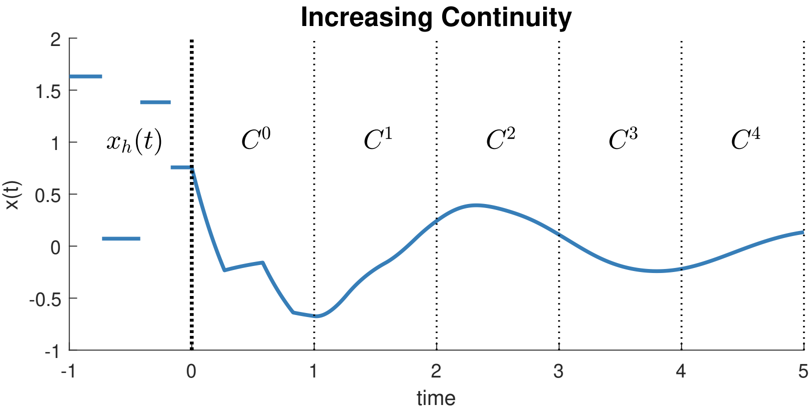
Degree d, dynamics degree $\tilde{d} = d + \max(\lfloor \deg f/2 \rfloor, \deg g - 1)$

Bounds: $p_d^* \geq p_{d+1}^* \geq \ldots \geq p_r^* = P_r^* \geq P^*$

Measure	$\mu_p(t, x)$	$\mu(t, x)$
PSD Size	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(1+n)^{6\tilde{d}}$ or $\tilde{d}^{4(n+1)}$

Propagation of Continuity



$$x'(t) = -2x(t) - 2x(t-1)$$

Continuity increases every τ_r time steps

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lfloor \deg f/2 \rfloor$

Bounds: $p^*_d \geq p^*_{d+1} \geq \ldots = p^* \geq P^*$

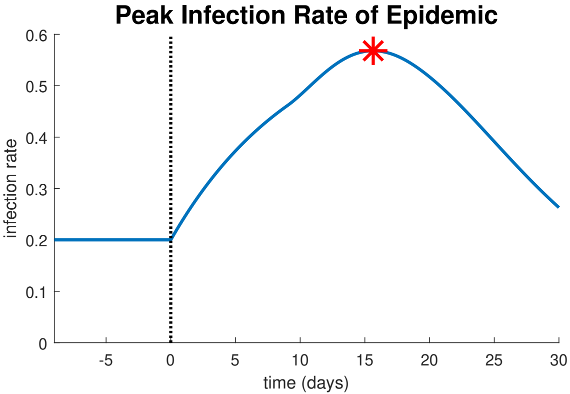
Size of Moment Matrices Peak Estimation

Measure:	μ_0	μ^p	μ_h
Size:	$\binom{n+d}{d}$	$\binom{n+1+d}{d}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Measure:	$\bar{\mu}_0$	$\bar{\mu}_1$	ν
Size:	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{2n+1+\tilde{d}}{\tilde{d}}$	$\binom{n+1+\tilde{d}}{\tilde{d}}$

Timing scales approximately as $(2n+1)^{6\tilde{d}}$ or $\tilde{d}^{4(2n+1)}$

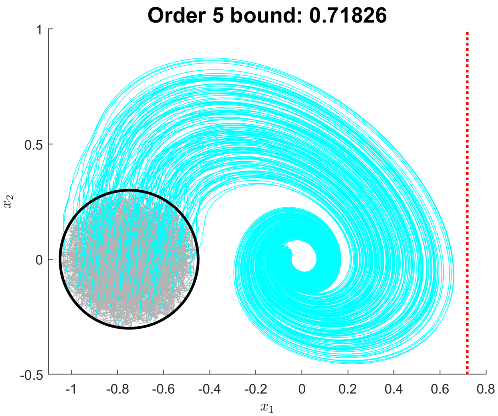
SIR Peak Estimation Example



Upper bound $I_{max} \geq 56.9\%$ with order 3 LMI

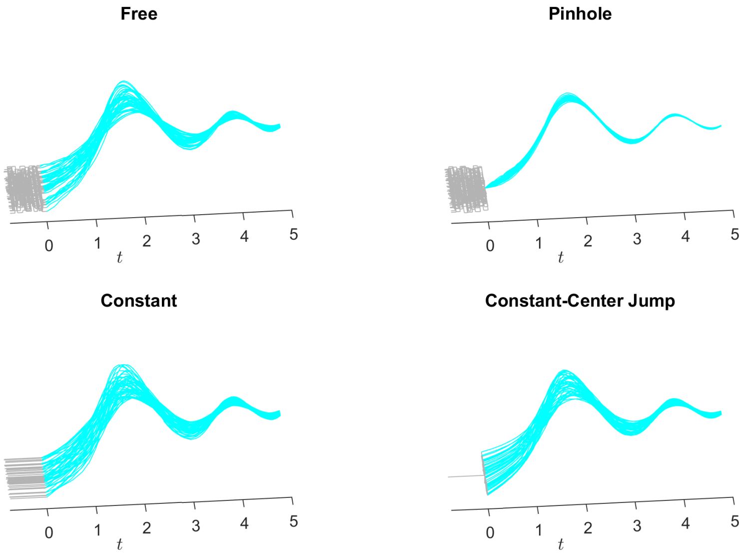
Recovery: $t_* = 15.6$ days, $(S^*, I^*) = (56.9\%, 5.61\%)$

Time-Varying System



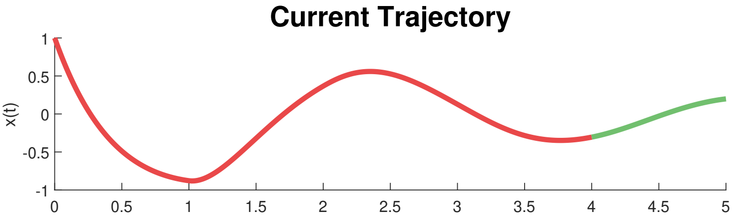
Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Time-Varying Histories



History restrictions and trajectories of system

Joint+Component Consistency

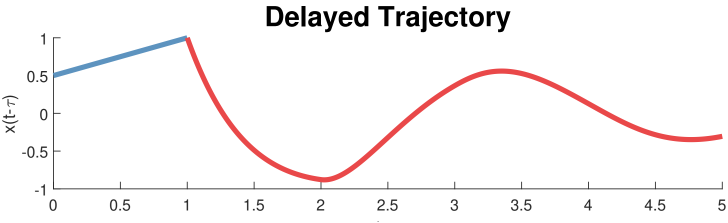


(t, x_0) marginal of $\bar{\mu}$

For all test functions $\phi_0 \in C([0, T] \times X)$

$$\begin{aligned}\langle \phi_0(t, x_0), \bar{\mu} \rangle &= \int_0^T \phi_0(t, x(t \mid x_h))dt \\ &= \left(\int_0^{T-\tau} + \int_{T-\tau}^{T} \right) \phi_0(t, x(t \mid x_h))dt \\ &= \langle \phi_0(t, x), \nu_0 + \nu_1 \rangle\end{aligned}$$

Joint+Component Consistency (cont.)



(t, x_1) marginal of $\bar{\mu}$

For all test functions $\phi_1 \in C([0, T] \times X)$

$$\begin{aligned} \langle \phi_1(t, x_1), \bar{\mu} \rangle &= \int_0^T \phi_1(t, x(t-\tau \mid x_h))dt \\ &= \int_{-\tau}^{T-\tau} \phi_1(t+\tau, x(t \mid x_h))dt \\ &= \int_{-\tau}^{0} \phi_1(t+\tau, x_h(t))dt + \langle \phi_1(t+\tau, x), \nu_0 \rangle \end{aligned}$$

Joint+Component Experiment

Table 1: Objective values for Flow experiment

degree d	1	2	3	4	5
Joint+Component	1.25	1.223	1.1937	1.1751	1.1636
Standard	1.25	1.2183	1.1913	1.1727	1.1630

Table 2: Time (seconds) to obtain SDP bounds in Table 1

degree d	1	2	3	4	5
Joint+Component	0.782	0.991	5.271	31.885	336.509
Standard	0.937	1.190	9.508	105.777	552.496

Bonus: Measure Background

Measures

Nonnegative Borel Measure μ

Assigns each set $A \subseteq X$ a 'size' $\mu(A) \geq 0$ (Measure)

Mass $\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

$\mu \in \mathcal{M}_+(X)$: space of measures on X

$f \in C(X)$: continuous function on X

Pairing by Lebesgue integration $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$

Dirac Delta Measure

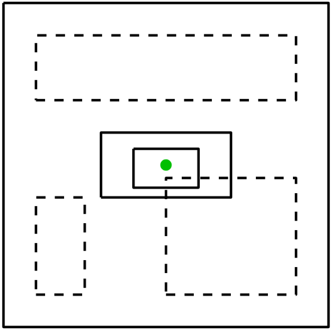
Dirac delta $\delta_{x'}(A) = \begin{cases} 1 & x' \in A \\ 0 & x' \notin A \end{cases}$

Probability:

$\delta_{x'}(X) = 1, \ \langle f(x), \delta_{x'} \rangle = f(x')$

$\mu(A) = 1$: Solid Box

$\mu(A) = 0$: Dashed Box



Atomic Measure

Rank-1 atomic measure

$$\mu = c\delta_{x'} \qquad c > 0$$

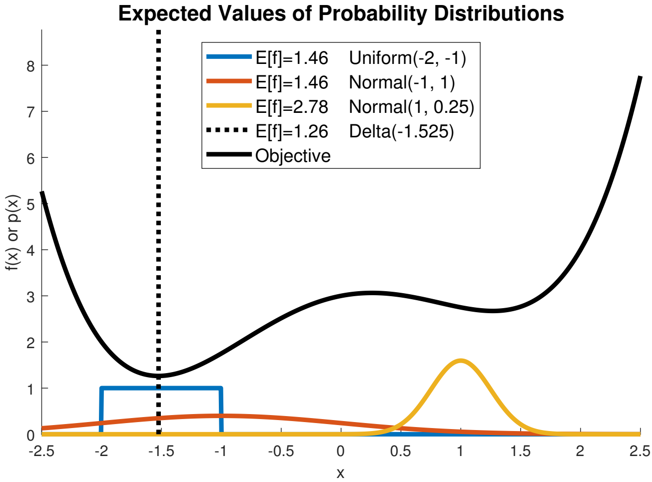
Rank-2 atomic measure

$$\mu = c_1\delta_{x'_1} + c_2\delta_{x'_2} \qquad c > 0,\ x'_1 \neq x'_2$$

Rank-r atomic measure

$$\mu = \sum_{i=1}^{r} c_i\delta_{x'_i} \qquad c > 0,\ \{x'_i\}_{i=1}^{r}\text{distinct}$$

Example of Measure Optimization

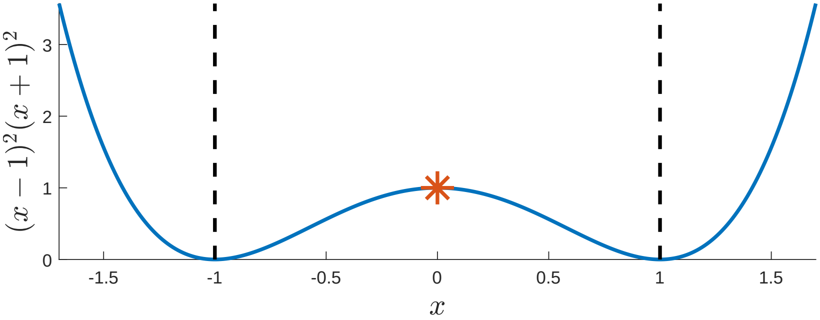


Optimum $\mathbb{E}_{\mu}[f] = \langle f, \mu \rangle$ at $\mu = \delta_{x^*}$

Measure Optimization

Nonconvex problems could be convex in measures

$$\min_{x \in K} p(x) \to \min_{\mu \in \mathcal{M}_+(K)} \langle p, \mu \rangle, \quad \langle 1, \mu \rangle = 1$$



$$f(\tfrac{1}{2}(1 + (-1))) = 1, \text{ but } \tfrac{1}{2}(f(1) + f(-1)) = 0$$

Bonus: Approximating Measure LPs

Need for Approximation

Measure LPs are infinite-dimensional

Linear Matrix Inequality: convex problem

$$\max_y b^T y \qquad C + \sum_{i=1}^{m} A_i y_i \geq 0$$

Solve LMIs through (interior point, ADMM, etc.)

Approximate infinite LPs by finite-dimensional LMIs

Moments

Monomial $x^\alpha = \prod_i x_i^{\alpha_i}$ for power $\alpha \in \mathbb{N}^n$

Degree $|\alpha| = \sum_i \alpha_i$

α-moment of measure $y_\alpha = \langle y_\alpha, \mu \rangle$

Measure uniquely described by infinite set $\{y_\alpha\}_{\alpha \in \mathbb{N}^n}$

When does a sequence $\{y_\alpha\}_{\alpha \in \mathcal{A}}$ correspond to a measure μ?

Linear Functional

Linear Functional polynomial $\rightarrow$ moments

$$f(x) \rightarrow \int_X f(x) d\mu = \int_X \sum_\alpha f_\alpha x^\alpha d\mu = \sum_\alpha f_\alpha y_\alpha$$

Bivariate Example

$$2 + x_1 x_2 - 3x_1^2 + x_1 x_2^3 \rightarrow 2 + y_{11} - 3y_{20} + y_{13}$$

Moment Matrices

Squares $f(x)^2$ are nonnegative (real)

$f(x)^2 \geq 0$ implies that $\langle f(x)^2, \mu \rangle \geq 0 \quad \forall f \in \mathbb{R}[x]$:

$$\langle f(x)^2, \mu \rangle = \int_X \sum_{\alpha,\beta} (f_\alpha x^\alpha)(f_\beta x^\beta) d\mu = \int_X \sum_{\alpha,\beta} (f_\alpha f_\beta x^{\alpha+\beta}) d\mu \geq 0$$

Moment matrix $\mathbb{M}[y] \succeq 0$ has $\mathbb{M}[y]_{\alpha,\beta} = y_{\alpha+\beta}$

$$\langle f(x)^2, \mu \rangle = \mathbf{f}^T \mathbb{M}[y] \mathbf{f} \geq 0$$

Moment Matrix Example

Moments up to degree $2 \times 2 = 4$

$$\mathbb{M}_2[y] = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{11} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}$$

Localizing Matrices

μ supported on set $K = \{x \mid g_i(x) \geq 0, i = 1 \ldots N\}$

$g_i(x)f(x)^2 \geq 0$ implies that $\langle g_i(x)f(x)^2, \mu \rangle \geq 0$

$$\langle g_i(x)f(x)^2, \mu \rangle = \int_X \sum_{\alpha,\beta,\gamma} (f_\alpha f_\beta g_\gamma x^{\alpha+\beta+\gamma}) d\mu \geq 0$$

Localizing matrix $\mathbb{M}[g_i m] \succeq 0$ has $\mathbb{M}[g_i m]_{\alpha,\beta} = \sum_\gamma g_\gamma m_{\alpha+\beta+\gamma}$

$$\langle g_i(x)f(x)^2, \mu \rangle = \mathbf{f}^T \mathbb{M}[g_i y] \mathbf{f} \geq 0$$

Moment-SOS Hierarchy

Polynomial optimization problem example :

$$p^* = \max_{x \in K} p(x) = \max_{\mu \in \mathcal{M}_+(K)} \langle p(x), \mu \rangle, \quad \mu(K) = 1$$

Keep moments up to degree d:

$$p_d^* = \max_y \ \sum_{|\alpha| \leq 2d} p_\alpha m_\alpha$$

$$\mathbb{M}_d[y], \ \mathbb{M}_{d-\deg(g_i)}[g_i y] \succeq 0$$

Finite-dimensional SDP: $\mathbb{M}_d[y]$ has size $\binom{n+d}{d}$

Bounds $p_d^* \geq p_{d+1}^* \geq p_{d+2}^* \ldots$ converge to p^* as $d \to \infty$

Approximation Pipeline

1. Trajectory Program
2. Measure LP
3. Moment LMI

Increase degree d of LMI to get better bounds

Prove conditions under which $\lim_{d \to \infty} p_d^* \to p^* = P^*$