Risk Analysis of Stochastic Processes using Linear Programming

Jared Miller

April 11, 2024

L2S: H-CODE Series

AUTOMATIC CONTROL LABORATORY

ETH zürich

Collaborators

Didier Henrion

Ashkan Jasour

Niklas Schmid

Roy S. Smith

Mario Sznaier

Matteo Tacchi

Motivation

Power System Example

Sources of randomness[1]:

- Thermal noise
- Measurement errors
- Unknown demand
- Intermittent wind/solar
- Extreme weather

Need to have reliable grid operation

[1] Bienstock, Daniel. Electrical transmission system cascades and vulnerability: an operations research viewpoint. Society for Industrial and Applied Mathematics, 2015.

Power System Failure

Comparatively small-scale: a building

Power System Failure

Large-scale: City of Houston, TX (Feb. 7 and 16, 2021)

NASA Earth Observatory/Joshua Stevens

Failure Modes

Overcurrent, overvoltage, voltage collapse, short circuit, etc.

Machines can safely pull large currents for short times

AC/DC converters have hard current limits

Risk Analysis Tasks

Different settings require different notions of risk.

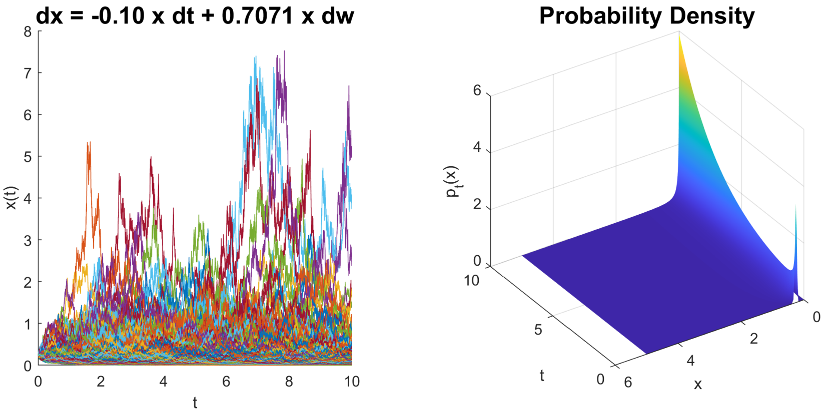
This seminar will discuss three risk quantifiers:

1. Probability of Unsafety
2. Instantaneous Risk
3. Time-Windowed Risk

Stochastics Background

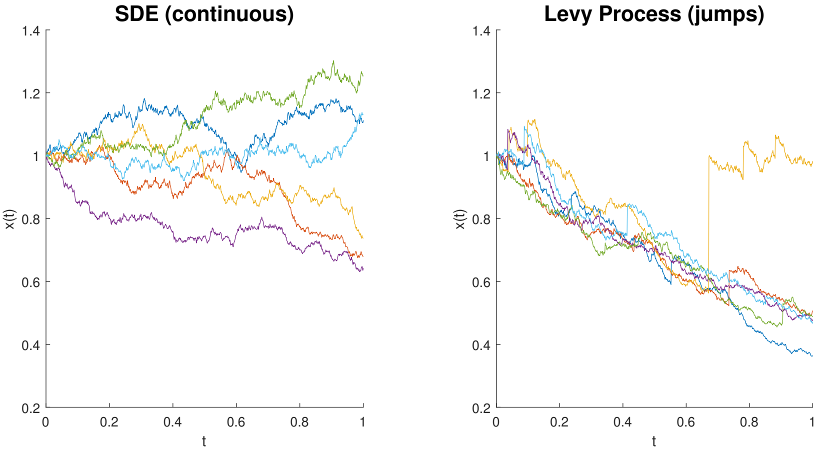
Stochastic Process

A collection of time-indexed probability distributions $\{\mu_t\}$



SDE: $dx = f(t,x)dt + g(t,x)dw$ (Itô)

Stochastic Process Examples



Geometric Brownian motion (left), Merton jump diffusion (right)

Questions to ask

Given a state function $p(x)$ (e.g. height, current, voltage)

Bound the following quantities along stochastic trajectories:

- Probability of entering unsafe set
- Mean of p
- 90% quantile of p
- Mean value above 90% quantile of p
- Other risk measures of p

Desired algorithm properties

Risk analysis problems are generally nonconvex

What we want in an algorithm:

Convex	single optimal solution
Tight	same objective value
Tractable	can be solved/approximated by computers
Bounded	have error bounds/sidedness
Bisection-free*	only solve a single problem

Our approach: infinite-dimensional LP in measures/functions

scenario approach: asymptotic/bounds in prob., no sidedness

Generator (Incremental Expected Change)

Generator $\mathcal{L}$ of process ($\forall v \in \mathrm{dom}(\mathcal{L}) = \mathcal{C}$):

$$\mathcal{L}_\tau v = \lim_{\tau' \to \tau} \left(\mathbb{E}[v(t + \tau', x) \mid \mu_{t+\tau'}] - v(t, x) \right) / \tau'$$

For all solutions $\{\mu_t\}_{t=0}^{T}$ (with $x(t) \sim \mu_t$) following $\mathcal{L}$, $\forall v \in \mathcal{C}$:

$$\mathbb{E}_{x \sim \mu_T}[v(T, x)] = \mathbb{E}_{x \sim \mu_0}[v(0, x)] + \int_{t=0}^{T} \mathbb{E}_{x \sim \mu_t}[\mathcal{L}_0 v(t, x)] dt$$

End = Start + Accumulated Change (in expectation)

Examples of Generators

Discrete-time Markov Process ($\mathcal{C} = C([0, T] \times X)$)

$$X_{t+\tau} = F(t, X_t, \omega_t), \qquad \omega_t \sim \xi \text{ (sampled)}$$
$$\mathcal{L}_\tau v = \left(\int_\Omega v\left(t+\tau, F(t, x, \omega)\right) d\xi(\omega) - v \right) / \tau$$

Stochastic Differential Equation ($\mathcal{C} = C^{1,2}([0, T] \times X)$)

$$dx = f(t, x)dt + g(t, x)dW,$$
$$\mathcal{L}_0 v = \partial_t v + f \cdot \nabla_x v + g^T(\nabla^2_{xx} v)g/2$$

Others: Lévy processes, hybrid, switching, time-delay

Stochastic Safety

Safety Problem

Hazardous unsafe set X_u present:

- The ground (when flying)
- Overcurrent
- Other cars on road
- Temperature Violation

Estimate probability of entering X_u

Iceland Monitor

Probability of Unsafety

Find probability of unsafety starting at X_0:

$$
\begin{aligned}
P^*(t_0, x_0) = \sup_{t^* \in [t_0, T]} \quad & \text{Prob}_{\mu_{t^*}}[x \in X_u] \\
\text{s.t.} \quad & x(t) \text{ follows } \mathcal{L} \qquad \forall t \in \min(t^*, \tau_X) \\
& x(0) \in X_0
\end{aligned}
$$

Worst-case over X_0: $P^*(t_0, X_0) = \sup_{x_0 \in X_0} P^*(t_0, x_0)$

τ_X is exit time distribution (leaving X)

Stochastic Barrier Functions

Proof of γ-probability safety for $x(0) \in X_0$ [2]

$$
\begin{aligned}
B^*(x) = \underset{B \in \mathcal{C}}{\text{find}}\ & B(x) \leq \gamma && \forall x \in X_0 \\
\text{s.t.}\ & B(x) \geq 1 && \forall x \in X_u \\
& B(x) \geq 0 && \forall x \in X \\
& \mathcal{L}B(x) \leq 0 && \forall x \in X
\end{aligned}
$$

Requires bisection on γ, inconclusive if γ fails (truncations)

[2] Prajna, Stephen, Ali Jadbabaie, and George J. Pappas. "A framework for worst-case and stochastic safety verification using barrier certificates." IEEE Transactions on Automatic Control 52.8 (2007): 1415-1428.

How do we solve infinite LPs?

Discretization necessary to solve on computer

More complexity: more accurate solutions

Method	Increasing Complexity
Gridding	# Grid Points
Basis Functions	# Functions
Random Sampling	# Samples
⋆ Sum-of-Squares (SOS)	Polynomial Degree
Your Favorite Method	Some Accuracy Parameter

Runtime usually exponential in dimension, complexity

Infeasibility: unsolvable problem or not enough compute?

Probability of Unsafety

Minimize probability γ, use time-dependent function $v(t, x)$

$$
\begin{aligned}
P(t_0, X_0) = \inf_{\gamma \in \mathbb{R}} \gamma & \\
\text{s.t.} \quad & \gamma \geq v(0, x) && \forall x \in X_0 \\
& \mathcal{L}v(t, x) \leq 0 && \forall (t, x) \in [t_0, T] \times X \\
& v(t, x) \geq 0 && \forall (t, x) \in [t_0, T] \times X \\
& v(t, x) \geq 1 && \forall (t, x) \in [t_0, T] \times X_u \\
& v \in \mathcal{C}
\end{aligned}
$$

$P(t, X_0) = P^*(t, X_0)$ under compactness, regularity

Averaged Probability of Safety

Average unsafe probability over initial distribution μ_0

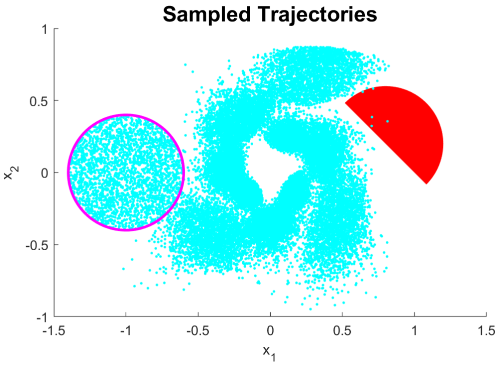
$$
\begin{aligned}
J^*(t_0, \mu_0) = \inf \int_X & v(t_0, x_0) d\mu_0(x_0) \\
\text{s.t.} \quad & \mathcal{L}v(t,x) \leq 0 && \forall (t,x) \in [t_0, T] \times X \\
& v(t,x) \geq 0 && \forall (t,x) \in [t_0, T] \times X \\
& v(t,x) \geq 1 && \forall (t,x) \in [t_0, T] \times X_u \\
& v \in \mathcal{C}
\end{aligned}
$$

Feasible solutions satisfy $v(t_0, x_0) \geq P^*(t_0, x_0)$

L_1 convergence $v \to P^*$ under same conditions

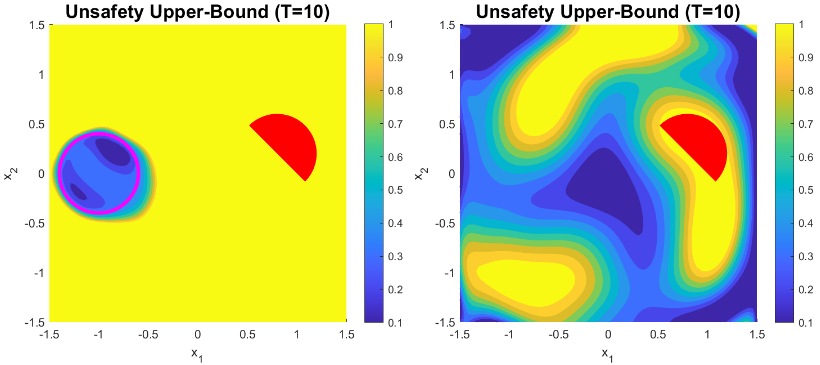
Discrete-Time System Example

$$x_+ = \begin{bmatrix} -0.3x_1 + 0.8x_2 + x_1 x_2 \lambda/4 \\ -0.9x_1 - 0.1x_2 - 0.2x_1^2 \end{bmatrix}, \qquad \lambda \in \mathcal{N}(0,1)$$



Unsafe probabilities $R_0 = 0 :\leq 7.052\text{e}{-4}$, $R_0 = 0.4 :\leq 0.4017$

Risk Contours

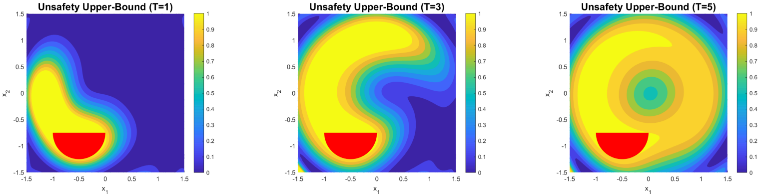


(a) Disc X_0 : $\mathrm{Prob} \leq 0.4017$ **(b)** Averaged unsafety contour

Risk contours (upper-bounds) with poly. deg $v(t, x) = 12$

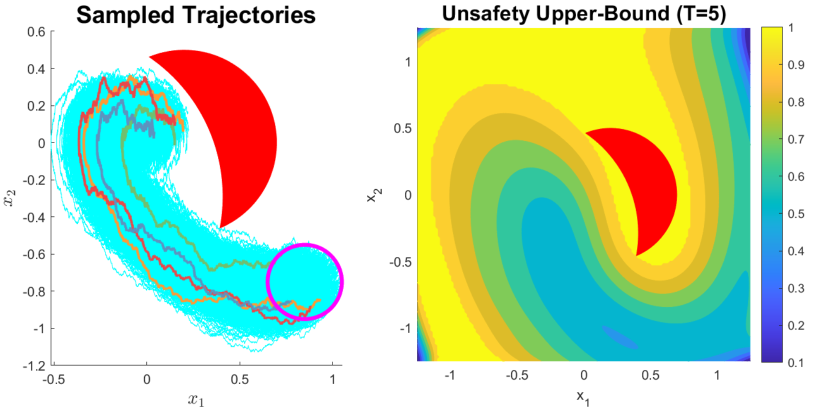
Risk Contour Evolution

Swept unsafe regions as T increases



Risk propagation of $dx = [-x_2; x_1]dt + [0; 0.1]dW$

Not Just Convex Obstacles!



Instantaneous Risk Estimation

Risk of a Distribution

Desired State function $p(x)$ (e.g., current, height, speed)

Pushforward $p_\#$: samples satisfy $p(x(t)) \sim p_\# \mu_t$

Some properties of $p_\# \mu_t$:

- Mean
- 90% Value-at-Risk (quantile)
- 90% Conditional Value-at-Risk
- Essential Supremum

Choose risk R from the above list, consider $R(p_\# \mu_t)$

Chance-Peak Problem

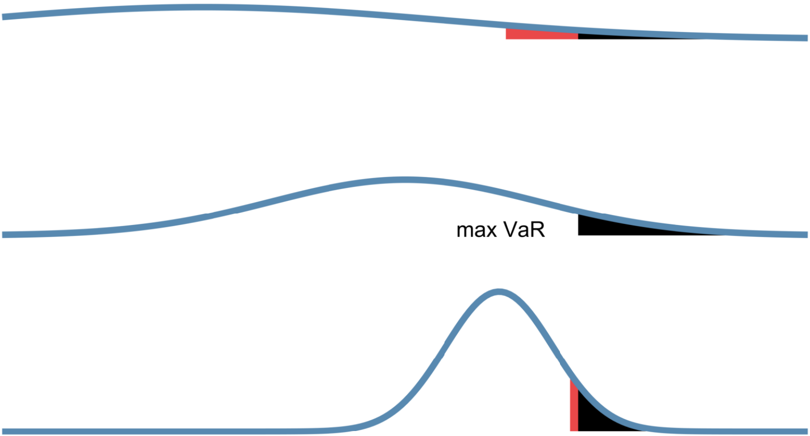
What is the maximum risk R along the stochastic trajectory?

$$
\begin{aligned}
P^* = \sup_{t^* \in [0, T]} \quad & R(p_\# \mu_{t^*}) \\
\text{s.t.} \quad & x(t) \text{ follows } \mathcal{L} \qquad \forall t \in [0, \min(t^*, \tau_X)] \\
& x(0) \sim \mu_0 \text{ (or } x(0) \in X_0)
\end{aligned}
$$

Quantifies safety: greater risk could mean more unsafe

Maximal Value at Risk (VaR)

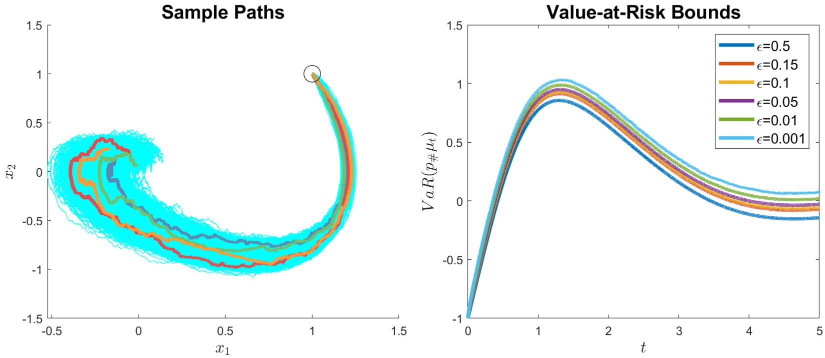
Maximize Value at Risk (Quantile Statistic) in time



Red + Black areas = 10% probability

Value-at-Risk Example (Monte Carlo)

50,000 samples with $T = 5, \ \Delta t = 10^{-3}$



VaR of $p = -x_2$ along $dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$

A brief interlude about measures

Why Talk about Measures?

Another source of intuition when forming programs

Measures LPs are (weakly) dual to function LPs

- strong duality under mild conditions
- solutions come ‘for free’ with discretization

What are Measures?

Nonnegative Borel Measure $\mu : \text{Set}(X) \to \mathbb{R}_+$ (σ-algebra)

$\mu \in \mathcal{M}_+(X)$: space of nonneg. measures on X

$f \in C(X)$: continuous function on X

Pairing by Lebesgue integration:

$$\langle f, \mu \rangle = \int_X f(x) d\mu(x)$$

$\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

Product measure $\mu_1 \otimes \mu_2 \in \mathcal{M}_+(X_1) \times \mathcal{M}_+(X_2)$

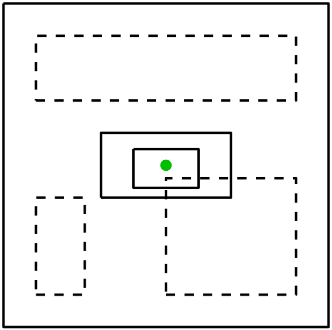
Dirac Delta Example

Dirac delta $\delta_{x=x'}(A) = \begin{cases} 1 & x' \in A \\ 0 & x' \notin A \end{cases}$

Probability measure: $\delta_{x=x'}(X) = 1$

$\mu(A) = 1$: Solid Box

$\mu(A) = 0$: Dashed Box



x' is the green dot

Notation: Time 0 is $\delta_{t=0} = \delta_0$

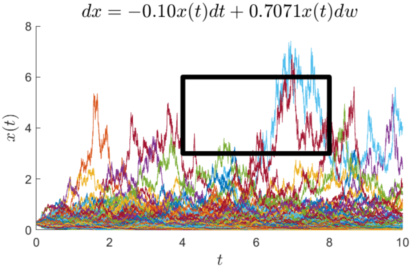
Occupation Measures (stochastic)

μ: stochastic kernel from $\{\mu_t\}$

Set $\mapsto$ Avg. time spent in the set

Average: μ_0 and stoch. dynamics

Averaged value of $v \in \mathcal{C}$:
$\langle v, \mu \rangle = \int_0^T \mathbb{E}_{x \sim X_t}[v(t,x)]dt$



Box: set in (t, x)

Martingale Relation

End = Start + Accumulated Change (in $\mathbb{E}$)

$$\forall v \in \mathcal{C}: \ \mathbb{E}[v(t+s,x) \mid \mu_{t+s}] = \mathbb{E}[v(t,x) \mid \mu_t] + \int_{t'=t}^{t+s} \mathbb{E}[\mathcal{L}v(t',x) \mid \mu_{t'}]dt'$$

Relation between measures (μ_t, μ_{t+s}, μ) for all $v \in \mathcal{C}$

$$\langle v(t+s,x), \mu_{t+s}(x)\rangle = \langle v(t,x), \mu_t(x)\rangle + \langle \mathcal{L}v, \mu\rangle$$

Compress notation using adjoint $\mathcal{L}^\dagger$ (implicitly express $\forall v$)

$$\mu_{t+s} = \mu_t + \mathcal{L}^\dagger \mu$$

Back to the regularly scheduled
instantaneous risk estimation

Mean Maximization

When R is the mean, can solve an infinite LP[3]:

$$\begin{aligned} p^* = \ & \sup \quad \langle p(x), \mu_\tau \rangle \\ \text{s.t.} \quad & \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\ & \mu, \mu_\tau \in \mathcal{M}_+([0, T] \times X) \end{aligned}$$

Instance of a stochastic Optimal Control Program[4]

(μ_τ^*, μ^*) is feasible with $P^* = \langle p(x), \mu_\tau^* \rangle \leq p^*$

$P^* = p^*$ if compactness, regularity properties hold

[3] Cho, Moon Jung, and Richard H. Stockbridge. "Linear programming formulation for optimal stopping problems." SICON 40.6 (2002): 1965-1982.

[4] Vinter, Richard B., and Richard M. Lewis. "The equivalence of strong and weak formulations for certain problems in optimal control." SICON 16.4 (1978): 546-570.

Value-at-Risk Bounds

VaR is nonconvex, nonsubadditive (unfriendly)

Concentration inequalities can upper-bound VaR

$$VaR_\epsilon(\nu) \leq \text{stdev}(\nu) r + \text{mean}(\nu)$$

Name	r value	Valid condition
Cantelli	$\sqrt{1/(\epsilon) - 1}$	ν probability distribution
VP	$\sqrt{4/(9\epsilon) - 1}$	ν unimodal, $\epsilon < 1/6$

(will talk about CVaR later)

Concentration-Bounded Chance-Peak

Apply concentration inequalities to get upper bound $P_r^* \geq P^*$

Objective upper-bounds VaR w.r.t. time-t^* distribution μ_{t^*}

$$P_r^* = \sup_{t^* \in [0,T]} r\sqrt{\langle p^2, \mu_{t^*}\rangle - \langle p, \mu_{t^*}\rangle^2} + \langle p, \mu_{t^*}\rangle$$

$$x \text{ follows } \mathcal{L}$$

$$x(0) \sim \mu_0$$

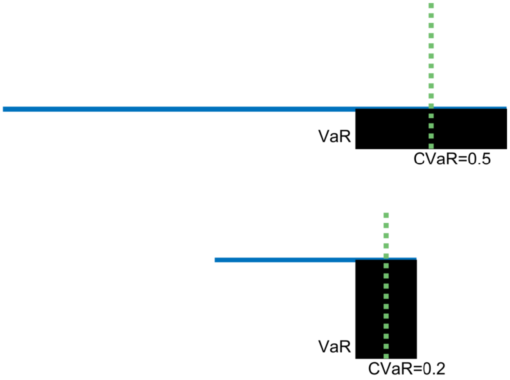
SOCP in measures for $p_r^* \geq P_r^* \geq P_{VAR}^*$ (3d SOC constraint)

Same constraints as mean-maximization, different objective

Conditional Value-at-Risk

CVaR: Average quantity above the Value-at-Risk

$$\text{CVaR}_\epsilon(\nu(\omega)) = (1/\epsilon)\int_{\omega \geq VaR_\epsilon(\nu)} \omega d\nu(\omega)$$



Uniform distributions with same VaR, different CVaR (70%)

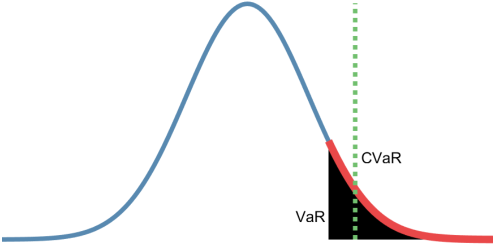
CVaR Linear Program

Measure LP to compute CVaR (with $\frac{d\psi}{d\nu} \leq \frac{1}{\epsilon}$)

$$\text{CVaR}_\epsilon(\nu) = \sup_{\psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R})} \quad \text{mean}(\psi)$$

$$\text{s.t.} \quad \epsilon\psi + \hat{\psi} = \nu$$

$$\langle 1, \psi \rangle = 1$$



VaR $= 1.2816$, CVaR$= 1.7550$, $\epsilon\psi \leq \nu$

CVaR Chance-Peak

Highest CVaR along stochastic trajectories

$$
\begin{aligned}
P_c^* = &\sup_{t^* \in [0, T]} \text{CVaR}_\epsilon(p_\# \mu_{t^*}) \\
\text{s.t.} \quad & x \text{ follows } \mathcal{L} \\
& x(0) \sim \mu_0
\end{aligned}
$$

Almost the same as VaR chance-peak, with $P_c^* \geq P^*$

CVaR Measure program

Add CVaR objective, constraints to chance-peak

$$
\begin{aligned}
p_c^* = \ &\sup \quad \text{mean}(\psi) \\
\text{s.t.} \quad & \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu \\
& \langle 1, \psi \rangle = 1 \\
& \epsilon\psi + \hat{\psi} = p_\# \mu_\tau \\
& \mu, \mu_\tau \in \mathcal{M}_+([0, T] \times X) \\
& \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R})
\end{aligned}
$$

Upper-bound $p_c^* \geq P_c^* \geq P^*$, LP in measures

Comparison of bounds

$P_r^* = p_r^*$ and $P_c^* = p_c^*$ if

1. Closure and boundedness conditions on $\mathrm{dom}(\mathcal{L})$
2. $[0, T] \times X$ compact (absorbing boundaries)
3. $p(x)$ is continuous (lower semicontinuous?)

$P^*_{\mathrm{Cantelli}} \geq P_c^*$ always, but $(P_c^*,\ P^*_{\mathrm{VP}})$ incomparable (so far)

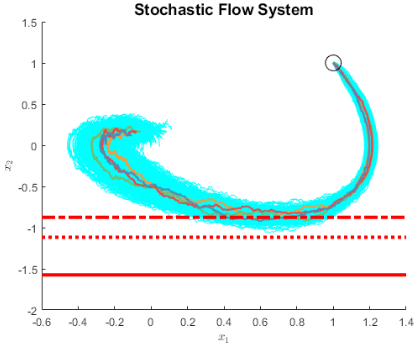
Empirically, degree-k moment LMIs satisfy $p^*_{\mathrm{Cantelli},k} \geq p^*_{c,k}$

Chance-Peak Examples

Two-State

Stochastic Flow (Prajna, Rantzer) with $T = 5,\ p(x) = -x_2$

$$dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

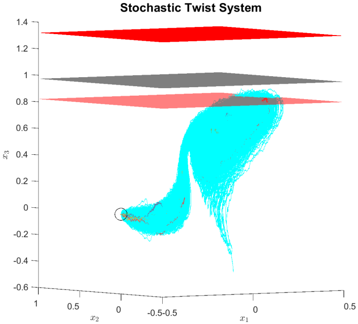


$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Three-State

Stochasic Twist system with $T = 5$, $p(x) = x_3$

$$dx = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} dw$$

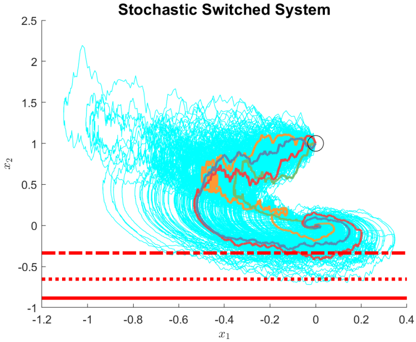


$d = 6$ (translucent=50%, gray=85% CVAR, solid=85% VP)

Two-State Switching

Switching subsystems at $T = 5$, $p(x) = -x_2$

$$dx = \left\{ \begin{bmatrix} -2.5x_1 - 2x_2 \\ -0.5x_1 - x_2 \end{bmatrix}, \begin{bmatrix} -x_1 - 2x_2 \\ 2.5x_1 - x_2 \end{bmatrix} \right\} dt + \begin{bmatrix} 0 \\ 0.25x_2 \end{bmatrix} dw$$

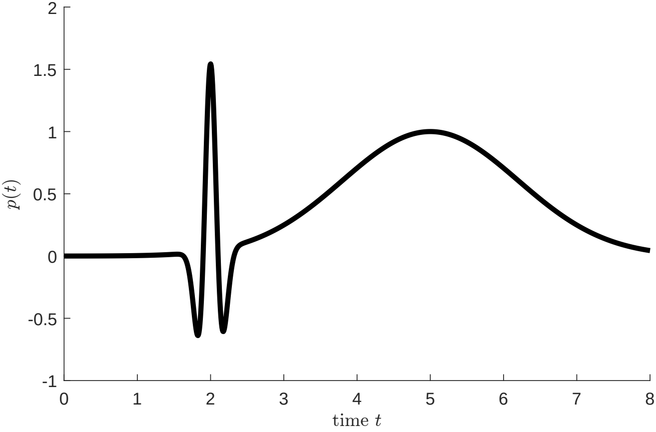


$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Time-Windowed Risk Estimation

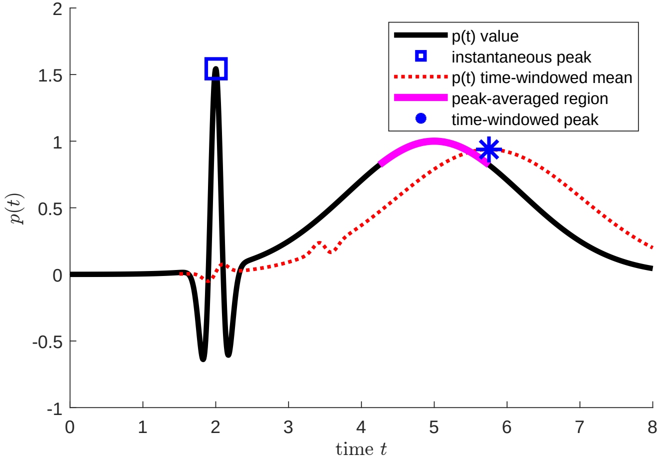
Time-Windowed Average Motivation: Signal

Oscillations near instanataneous peak ($t = 2$)



Time-Windowed Average Example

Instantaneous maximal risks may not give full picture



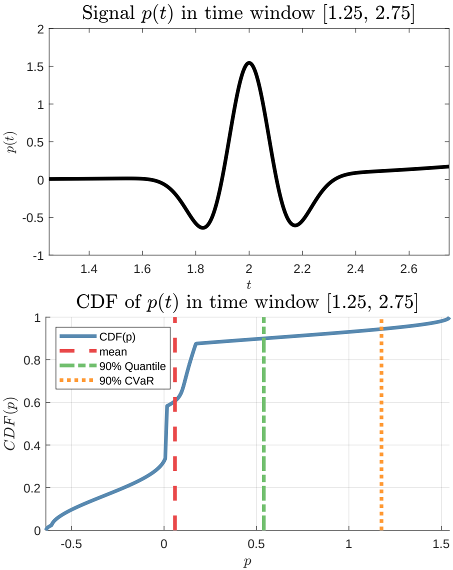
Large time-windowed avg. current on wire ≈ overheating

Time-Windowed Risk Example

Choose a time window h

Form a prob. dist. $\zeta(t)$ from $\{p(x(t'))\}_{t'=t-h}^{t}$

Analyze risk of $R(\zeta(t))$



Time-Windowed Risk Problem

Given R and h, choose optimal t^*, x_0^*:

$$\begin{aligned} P^* = \sup_{t^*,\ x_0^*} \quad & R\left(\frac{1}{h}\int_{t^*-h}^{t^*} p(x(t'))dt'\right) \\ \text{s.t.} \quad & x(t) \text{ follows } \mathcal{L} \quad \forall t \in [0, \min(t^*, \tau_X)] \\ & x(0) = x_0^* \\ & t^* \in [h, T],\ x_0^* \in X_0 \end{aligned}$$

Integral in objective collapses (marginalizes) time

Limits: Chance-peak as $h \to 0$, Risk-averse stopping as $h \to T$.

Augmented Time Coordinate

We can stick to ODE methods by adding a new time s

Two continuous times (t, s):

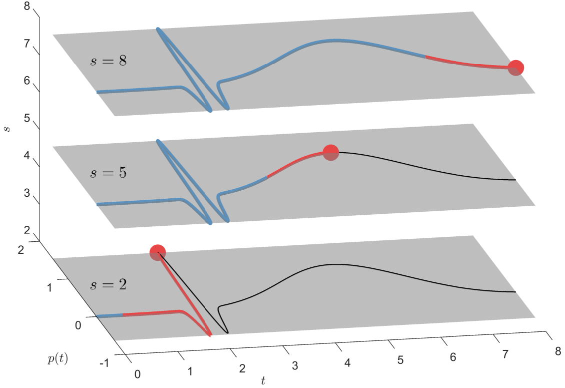
Active time	$t \in [0, T]$	$\dot{t} = 1$
Stopping time	$s \in [h, T]$	$\dot{s} = 0$

Temporal support sets $\Omega_\pm$:

$$\Omega_- : \quad t \in [0, s-h] \qquad\qquad \Omega_+ : \quad t \in [s-h, s]$$

Risk evaluated in Ω_+, similar process in discrete-time

Two Time Coordinates?



Curves $(t, p(t), s)$: time intervals $[0, s-h]$, $[s-h, s]$, $[s, T]$

Measures for Risk Estimation

Mean-type risk estimation measures (with constant state s)

$$\begin{aligned} \mu_0(s,x) &\in \mathcal{M}_+([h,T]\times X_0) && \text{Initial} \\ \mu_\tau(s,x) &\in \mathcal{M}_+([h,T]\times X) && \text{Terminal} \\ \mu_+(s,t,x) &\in \mathcal{M}_+(\Omega_+ \times X) && \text{Risk Occ.} \\ \mu_-(s,t,x) &\in \mathcal{M}_+(\Omega_- \times X) && \text{Past Occ.} \end{aligned}$$

Time-windowed risk evaluation: $\frac{1}{h}\int_{s-h}^{s} p(x(t'))dt' \to \frac{1}{h} p_{\#}\mu_+$

Time-Duplication Map

The last technical detail needed: a time-duplicating map φ

$$\varphi : (s, x) \mapsto (s, s, x)$$

For all test functions $\omega(s, t, x) \in C([h, T] \times [0, T] \times X)$

$$\langle \omega(s, t, x), \varphi_{\#}\mu_\tau(s, t, x) \rangle = \langle \omega(s, s, x), \mu_\tau(s, x) \rangle$$

Relaxed occupation measure of $\hat{\mathcal{L}}$: $(\mu_0, \varphi_{\#}\mu_\tau, \mu_+ + \mu_-)$

Time-Windowed Risk Estimation

Non-conservative infinite LP with generator $\hat{\mathcal{L}} : (\mathcal{L}, \dot{s} = 0)$

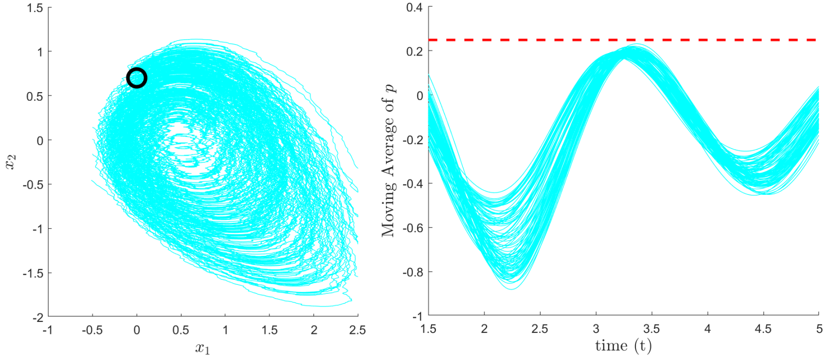
$$
\begin{aligned}
p^* = \ & \sup \quad \langle p, \mu_+ \rangle / h \\
& \text{s.t. } \varphi_\# \mu_\tau = \delta_0 \otimes \mu_0 + \hat{\mathcal{L}}^\dagger (\mu_- + \mu_+) \\
& \qquad \langle 1, \mu_0 \rangle = 1 \\
& \qquad \langle 1, \mu_+ \rangle = h \\
& \qquad \text{Mean-type time-windowed support constraints}
\end{aligned}
$$

Constraint $\langle 1, \mu_+ \rangle = h$ imposes that h time units elapse

CVaR modification: $\sup \text{mean}(\psi) : \ \epsilon \psi + \hat{\psi} = (p_\# \mu_+)/h$

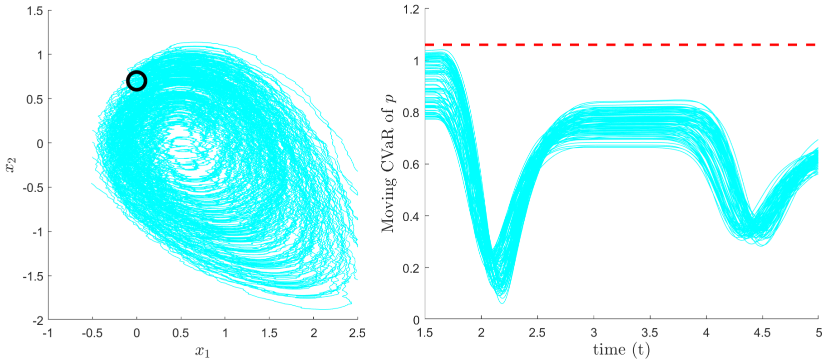
Time-Windowed Stoch. Mean Example ($h = 1.5$)

Instantaneous and time-windowed mean are separated ($p(x) = x_2$)



Time-Windowed Stoch. CVaR Example ($h = 1.5$)

Peak CVaR is close to peak instantaneous p (with $\epsilon = 0.15$)



Risk-Aware Control (overview)

Risk-Aware Control

Minimize in risk in objective/risk constraints (difficult)

Approximate

- Sample-Average Approximations
- Upper-bound CVAR
- Min-Max Approaches
- Boole's Inequality

Exact (up to structure)

- Distributional Robustness
- Dynamic Programming Principles (unconstrained)
- Nested Risk
- (Joint Chance Constraint)

Continuous-time: Pontryagin Max., not constructive

Talk to Gabriel Velho, Riccardo Bonalli, Benoît Bonnet-Weill

Take-aways

Conclusion

Risk estimation is important

Three problems: Unsafe Prob., Chance-Peak, Time-Window

Solved using infinite-dimensional LPs/SOCPs in measures

Certified outer-approximations of risk

Nice risk-aware control/tractable analysis is still open

Main References

Part 1: Jared Miller, Matteo Tacchi, Didier Henrion, and Mario Sznaier. *Unsafe Probabilities and Risk Contours for Stochastic Processes using Convex Optimization*, 2024. arXiv:2401.00815

Part 2: Jared Miller, Matteo Tacchi, Mario Sznaier, and Ashkan Jasour. *Peak Value-at-Risk Estimation of Stochastic Processes using Occupation Measures,* 2024. arXiv:2303.16064

Part 3: Jared Miller, Niklas Schmid, Matteo Tacchi, Didier Henrion, and Roy S. Smith. *Peak Time-Windowed Risk Estimation of Stochastic Processes*, 2024. arXiv:2404.06961

Thanks!

Bonus Slides

Main works

Miller, J., Tacchi, M., Henrion, D., Sznaier, M. (2024). Unsafe probabilities and risk contours for stochastic processes using convex optimization. arXiv:2401.00815.

Miller, J., Tacchi, M., Sznaier, M., Jasour, A. (2023). Peak Value-at-Risk Estimation for Stochastic Differential Equations using Occupation Measures. In 2023 62nd IEEE Conference on Decision and Control (CDC) (pp. 4836-4842). IEEE.

Miller, J., Tacchi, M., Sznaier, M., Jasour, A. (2023). Peak Value-at-Risk Estimation for Stochastic Stochastic Processes using Occupation Measures. arxiv:2303.16064.

Assumptions for Stochastic LPs

Assumptions used in all presented programs[5]:

1. Trajectories stop upon the first exit from X ($\tau_X \wedge T$).
2. The test function set $\mathcal{C} = \mathrm{dom}(\mathcal{L})$ satisfies $\mathcal{C} \subseteq C([t_0, T] \times X)$ with $1 \in \mathcal{C}$ and $\mathcal{L}1 = 0$.
3. The set $\mathcal{C}$ separates points and is multiplicatively closed.
4. There exists a countable set $\{v_k\} \in \mathcal{C}$ such that $\forall v \in \mathcal{C}$: $(v, \mathcal{L}v)$ is contained in the bounded pointwise closure of the linear span of $\{(v_k, \mathcal{L}v_k)\}$.

[5] Cho, Moon Jung, and Richard H. Stockbridge. "Linear programming formulation for optimal stopping problems." SICON 40.6 (2002): 1965-1982.

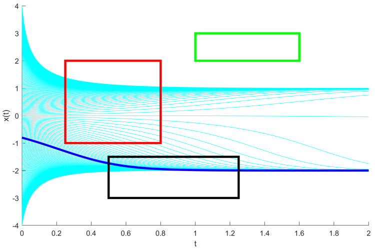
Occupation Measure (Deterministic)

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x + 2)(x - 1)$

Unsafe Probability using Measures

Maximize prob. $\langle 1, \mu_p \rangle$ of ending in X_u (with $\mu_p + \mu_c = \mu_\tau$)

$$
\begin{aligned}
p^*(t_0, X_0) = \sup \quad & \langle 1, \mu_p \rangle \\
\text{s.t.} \quad & \mu_p + \mu_c = \delta_{t_0} \otimes \mu_0 + \mathcal{L}^\dagger \mu \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu_0 \in \mathcal{M}_+(X_0) \\
& \mu, \ \mu_c \in \mathcal{M}_+([t_0, T] \times X) \\
& \mu_p \in \mathcal{M}_+([t_0, T] \times X_u)
\end{aligned}
$$

Relaxed occupation measure $(\mu_0, \mu_u + \mu_c, \mu)$,

Strongly dual to previous continuous-function program

SOS Expectation-Peak

$$
\begin{aligned}
d_{\mathbb{E}}^* = \min \quad & \int_X v(0,x) \; d\mu_0(x) && \text{(8a)} \\
\text{s.t.} \quad & -\mathcal{L}v(t,x) \in \Sigma[[0,T] \times X] && \text{(8b)} \\
& v(t,x) - p(x) \in \Sigma[[0,T] \times X] && \text{(8c)} \\
& v \in \mathbb{R}[t,x] && \text{(8d)}
\end{aligned}
$$

SOS Concentration-Peak

Second-order cone $\mathbb{L}^n : \{(u, q) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid q \geq \|u\|_2\}$

$$
\begin{aligned}
d_r^* = \min \quad & u_1 + 2u_3 + \int_{X_0} v(0, x_0) d\mu_0(x_0) && \text{(9a)} \\
\text{s.t.} \quad & -\mathcal{L}v(t, x) \in \Sigma[[0, T] \times X] && \text{(9b)} \\
& v(t, x) + u_1\, p^2(x) - 2\, u_2\, p(x) - p(x) && \text{(9c)} \\
& \qquad \qquad \in \Sigma[[0, T] \times X] \\
& ([u_1 + u_3, -(r/2), u_2], u_3) \in \mathbb{L}^3 && \text{(9d)} \\
& u \in \mathbb{R}^3, \ v \in \mathbb{R}[t, x]
\end{aligned}
$$

SOS CVaR-Peak

$$
\begin{aligned}
d_c^* = \min \quad & u + \int_X v(0,x)\ d\mu_0(x) && \text{(10a)}\\
\text{s.t.} \quad & -\mathcal{L}v(t,x) \in \Sigma[[0,T]\times X] && \text{(10b)}\\
& v(t,x) - w(p(x)) \in \Sigma[[0,T]\times X] && \text{(10c)}\\
& u + \epsilon\, w(q) - q \in \Sigma[p_{min}, p_{max}] && \text{(10d)}\\
& w(q) \in \Sigma[p_{min}, p_{max}] && \text{(10e)}\\
& u \in \mathbb{R}, v \in \mathbb{R}[t,x] && \text{(10f)}
\end{aligned}
$$

Time-Delay Approach (Bad, Don't Do This)

Embed as non-Markovian stochastic process:

$$
\begin{aligned}
P^* = \sup_{t^*,\ x_0^*} \quad & R\left(\beta(t^*)\right) \\
\text{s.t.} \quad & x(t) \text{ follows } \mathcal{L} \quad \forall t \in [0, \min(t^*, \tau_X)] \\
& d\beta = [p(x(t)) - p(x(t-h))](1/h)dt \\
& \beta(h) = (1/h) \int_0^h p(x(t'))dt' \\
& x(0) = x_0^* \\
& t^* \in [h, T],\ x_0^* \in X_0
\end{aligned}
$$

Could introduce relaxation gap, requires $2n + 2$ states

SOS Time-Window Mean

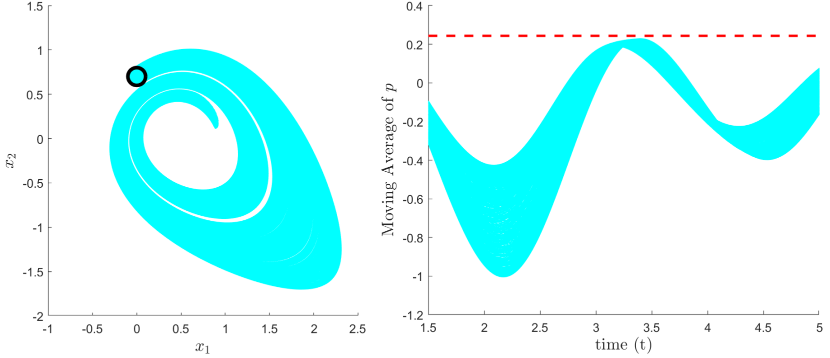
$$
\begin{aligned}
d_k^* = \ & \min_{v,\gamma,\xi} \gamma + h\xi && \text{(11a)} \\
\text{s.t.} \quad & \gamma - v(s, 0, x) \in \Sigma[[h, T] \times X_0] && \text{(11b)} \\
& v(t, t, x) \in \Sigma[[h, T] \times X]_{\leq 2k} && \text{(11c)} \\
& \xi - p(x)/h - \hat{\mathcal{L}} v(s, t, x) \in \Sigma[\Omega_+ \times X] && \text{(11d)} \\
& -\mathcal{L}_f v(s, t, x) \in \Sigma[\Omega_- \times X] && \text{(11e)} \\
& v \in \mathbb{R}[s, t, x] && \text{(11f)} \\
& \gamma, \xi \in \mathbb{R} && \text{(11g)}
\end{aligned}
$$

SOS Time-Window CVAR

$$
\begin{aligned}
d_k^* = \min_{v,\gamma,\xi,\beta,w} \quad & \gamma + h\xi + \beta && \text{(12a)} \\
\text{s.t.} \quad & \gamma - v(s,0,x) \in \Sigma[[h,T] \times X_0] && \text{(12b)} \\
& v(t,t,x) \in \Sigma[[h,T] \times X] && \text{(12c)} \\
& \xi - w(p(x))/h - \hat{\mathcal{L}} v(s,t,x) \in \Sigma[\Omega_+ \times X] && \text{(12d)} \\
& -\mathcal{L}_f v(s,t,x) \in \Sigma[\Omega_- \times X] && \text{(12e)} \\
& w(q),\ \epsilon w(q) + \beta \in \Sigma[[p_{\min}, p_{\max}]] && \text{(12f)} \\
& v \in \mathbb{R}[s,t,x] && \text{(12g)} \\
& w \in \mathbb{R}[q] && \text{(12h)} \\
& \gamma, \xi, \beta \in \mathbb{R} && \text{(12i)}
\end{aligned}
$$

Time-Windowed Mean Example ($h = 1.5$)

Instantaneous and time-windowed mean are separated ($p(x) = x_2$)



Time-Windowed CVaR Example ($h = 1.5$)

Peak CVaR is close to peak instantaneous p (with $\epsilon = 0.15$)