# Maximizing the Slice-Volume of Semialgebraic Sets

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# Volume Computation but Worse

# Flow of Presentation

Background of Volume, Slice-Volume problems

SOS-based algorithm to solve Slice-Volume

Complexity reductions to improve tractability

Stokes constraints to improve accuracy

# Background

# Volume vs. Slice-Volume

We have a set $L \in \mathbb{R}^n$ inside ball of radius $R < \infty$

**Volume (prior work):**

Find $V^* = \mathrm{Vol}_n(L)$

**Slice-Volume (ours):**

Find a direction $\theta$ and an affine offset $t$ to supremize

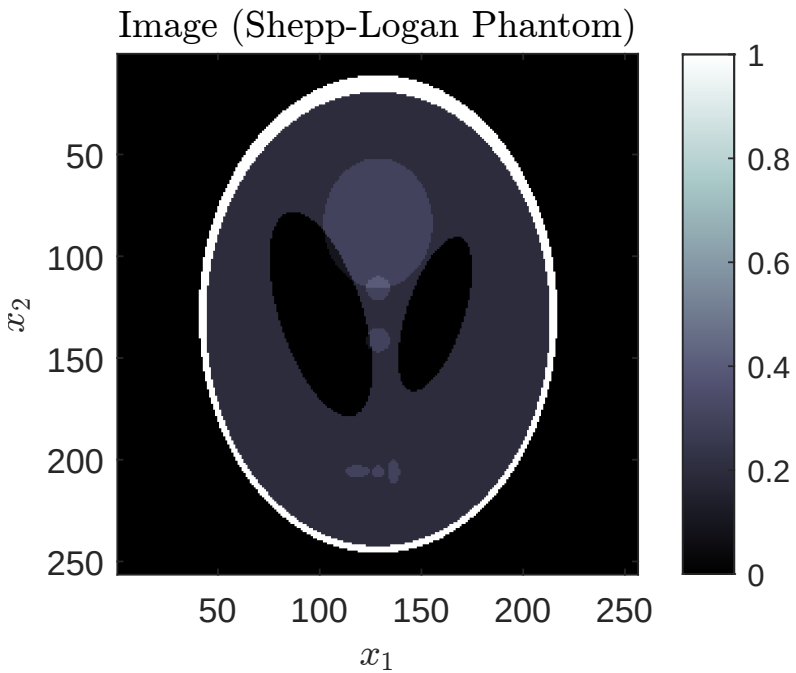
$$P^* = \sup_{\theta, t} \mathrm{Vol}_{n-1}\left((\theta \cdot x = t) \cap L\right)$$

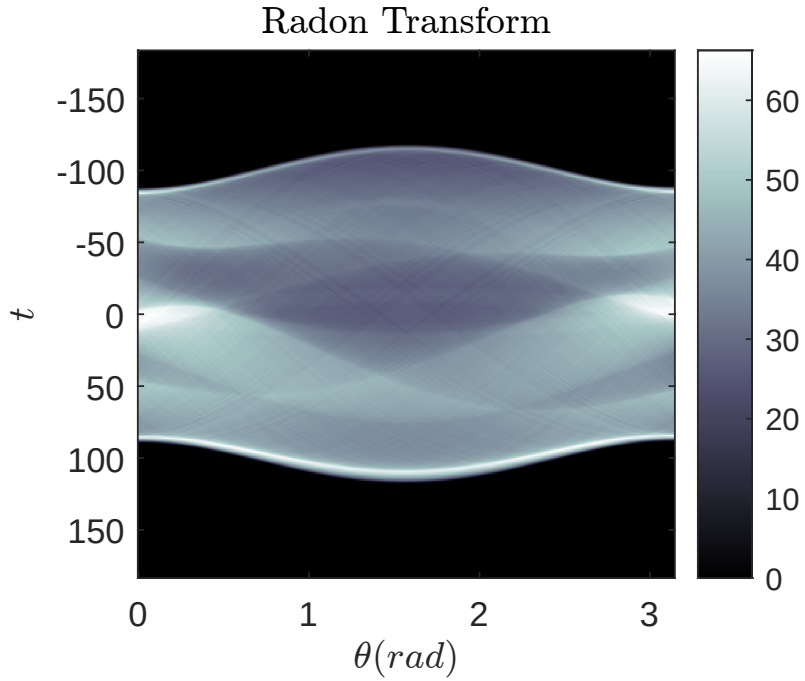
$$\theta \in S^{n-1}, \ t \in [-R, R].$$

# Slice-Volume: Radon Transform

Radon Transform of function $f$ (supported in $X$):

$$\mathcal{R}f(\theta, t) = \int_{(\theta \cdot x = t) \cap X} f(x)dx$$




Slice-Volume: Find $(\theta, t)$ to supremize $\mathcal{R}I_L$ (indicator of $L$)

# Slice-Volume: History

**Busemann-Petty problem (1956)** [1]

For convex $(L, T)$, does the statement

$$\forall \theta \in S^n : \mathrm{Vol}_{n-1}\left((\theta \cdot x = 0) \cap L\right) \geq \mathrm{Vol}_{n-1}\left((\theta \cdot x = 0) \cap T\right)$$

imply that $\mathrm{Vol}_n(L) \geq \mathrm{Vol}_n(T)$?

Busemann-Petty true in $n \leq 4$, false in $n \geq 5$.

Slice-Volume solved completely for polytopes[2]

[1] Busemann, Herbert; Petty, Clinton Myers (1956), "Problems on convex bodies", Mathematica Scandinavica, 4: 88–94

[2] K. Berlow, M.-C. Brandenburg, C. Meroni, and I. Shankar, "Intersection bodies of polytopes," Beiträge zur Algebra und Geometrie. Contributions to Algebra and Geometry, vol. 63, no. 2, pp. 419–439, 2022

# How to compute volume?

Volume computation is #P hard, even for polytopes

Some established methods include:

- Monte-Carlo Methods (doubly exponentially slow)
- Real algebraic geometry (rational periods)
- **Moment-SOS hierarchy**

# Volume Computation LP

Find volume of a set $L \subset X$ [3]

$$V^* = \inf \int_X w(x)dx \tag{1a}$$

$$w(x) \geq 0 \qquad \forall x \in X \tag{1b}$$

$$w(x) - 1 \geq 0 \qquad \forall x \in L \tag{1c}$$

$$w(x) \in C(X) \tag{1d}$$

Restrict $\geq 0$ (continuous) to your preferred Psätz ($\mathbb{R}[x]$).

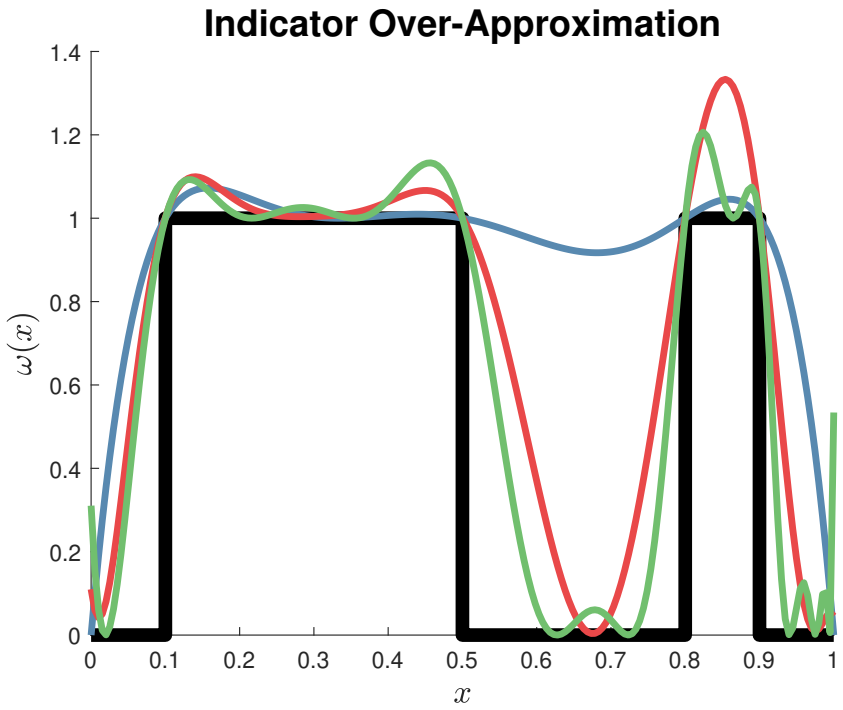
Convergence rate: $V_k^* - V^* \in O(k^{-z})$ for some $z > 0$ [4]

[3] D. Henrion, J. B. Lasserre, and C. Savorgnan, "Approximate volume and integration for basic semial- gebraic sets," SIAM review, vol. 51, no. 4, pp. 722–743, 2009

[4] C. Schlosser, M. Tacchi, and A. Lazarev, "Convergence rates for the moment-sos hierarchy," arXiv preprint arXiv:2402.00436, 2024.

# Volume Computation in Practice (Stokes later)

Volume of $L = [0.1, 0.5] \cup [0.8, 0.9]$ within $X = [0, 1]$



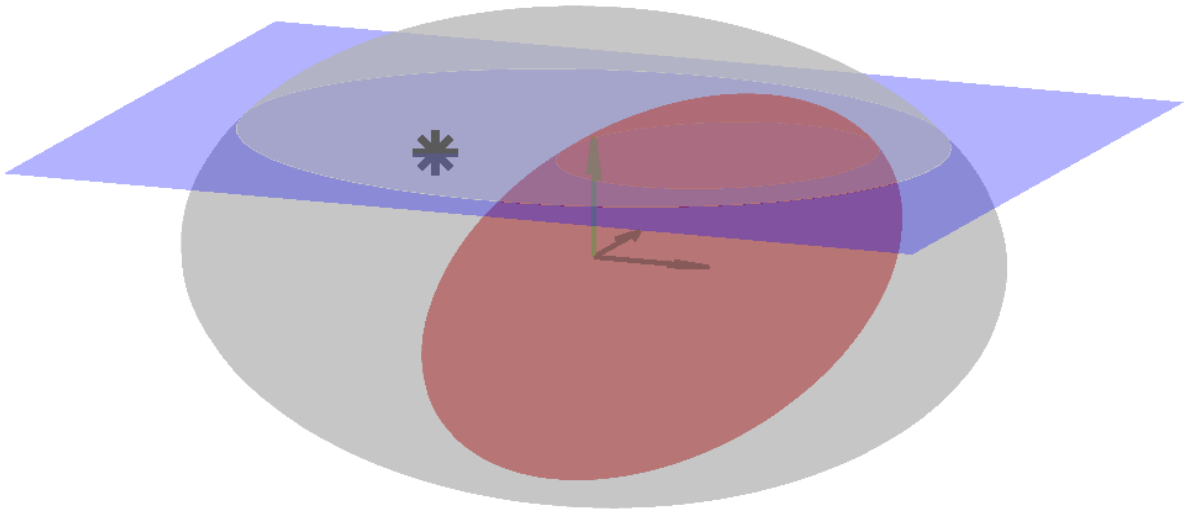

| Function | $k = 6$ | $k = 20$ | $k = 120$ | Truth |
|---|---|---|---|---|
| Volume (upper) | 0.9269 | 0.7472 | 0.6740 | 0.5 |

# Slice-Volume Variables

Slice-Volume is defined in terms of $(\theta, t)$

New members:

- $Z \in \mathbb{R}^{n \times (n-1)}$ (local coordinate frame)
- $y \in \mathbb{R}^{n-1}$ (local offset in frame defined by $Z$)



Global frame expression ($x \in B_R^n$): $x = \theta t + Zy$

# Support Sets

Domain of optimization variables $(\theta, t, Z)$:

$$\Omega = \{(\theta, t) \in S^n \times [-R, R]\}$$
$$\Omega_Z = \{(\theta, t, Z) \in S^n \times [-R, R] \times \mathbb{R}^{n\times(n-1)} \mid [\theta, Z] \in O(n)\}$$

Reparameterization of $B_R^n$ and $L$

$$\Psi = \{(\theta, t, Z, y) \in \Omega_Z \times B_R^{n-1}\}$$
$$\Psi_L = \Psi \cap \{(\theta t + Zy) \in L\}$$

# Slice-Volume Auxiliary Function

Volume computation had $w(x) \in C(X)$

Slice-volume has $w(\theta, t, Z, y) \in C(\Psi)$

Linear operator to integrate out $y$ coordinate:

$$\Lambda_R w(\theta, t, Z) = \int_{B_R^{n-1}} w(\theta, t, Z, y) dy$$

Association with radon transform $\mathcal{R}$:

$$\Lambda_R I_{\Psi_L}(\theta, t, Z) = \mathcal{R} I_L(\theta, t)$$

# Slice-Volume Program

Constrain that $w \geq I_{\Psi_L}$

$$
\begin{aligned}
p^* = \inf_{\gamma \in \mathbb{R}} \quad & \gamma && && \text{(2a)}\\
& \gamma \geq \Lambda_R w(\theta, t, Z) && \forall (\theta, t, Z) \in \Omega_Z && \text{(2b)}\\
& w(\theta, t, Z, y) \geq 1 && \forall (\theta, t, Z, y) \in \Psi_L && \text{(2c)}\\
& w(\theta, t, Z, y) \geq 0 && \forall (\theta, t, Z, y) \in \Psi && \text{(2d)}\\
& w(\theta, t, Z, y) \in C(\Psi). && && \text{(2e)}
\end{aligned}
$$

There is no relaxation gap (infimum) $p^* = P^*$

# SOS Considerations

Restrict $w(\theta, t, Z, y)$ to be polynomial

Good news: SOS convergence $p_k^* \to P^*$ (if $L$ Archimedean)

Bad news: Poor computational scaling

| Problem | variables | # parameters | PSD matrix size |
|---|---|---|---|
| Volume | $x$ | n | $\binom{n+k}{k}$ |
| Slice-Volume | $(\theta, t, Z, y)$ | $n^2 + 2n$ | $\binom{n^2+2n+k}{k}$ |

This is prohibitive for SDP methods to solve

# Complexity Reduction

# Complexity Reduction Overview

Three (interoperable) ways to reduce complexity:

1. Symmetry (separate from symmetries of $L$)
2. Algebraic structure
3. Topological properties ($n \in \{2, 3, 4, 8\}$)

(Stokes constraints improve accuracy, but increase complexity)

# Symmetry

Consider the constraint $(\theta t + Zy) \in L$ in $\Psi_L$

Discrete symmetry $(\theta, t, Z, y) \leftrightarrow (-\theta, -t, -Z, -y)$

Also $O(n-1)$ continuous symmetry:

$$\forall P \in O(n-1): \quad (\theta, t, Z, y) \leftrightarrow (\theta, t, ZP^T, Py)$$

Discrete: choose $w$ even

Continuous: harder, derive invariant SOS ring [5] (help?)

[5] K. Gatermann and P. A. Parrilo, "Symmetry groups, semidefinite programs, and sums of squares," Journal of Pure and Applied Algebra, vol. 192, no. 1-3, pp. 95–128, 2004

# Algebraic Structure

Support sets have equality constraints:

$$\|\theta\|_2^2 = 1 \qquad\qquad [\theta\ Z] \in O(n)$$

Use Gröbner basis reduction on constraints [6]

(need SAGBI/subduction to use with symmetry)

[6] P. A. Parrilo, "Exploiting structure in sum of squares programs," in 42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475), vol. 5. IEEE, 2003, pp. 4664–4669.

# Topological Properties ($n \in \{2, 4, 8\}$)

A manifold $\mathcal{M}$ is **parallelizable** if there is a continuous map from $x \in \mathcal{M}$ to a coordinate frame at $x$

Only spheres that are parallelizable[7]: $S^1, S^3, S^7$

Explicit (nonunique) parameterization for $Z$ in terms of $\theta$

$$[\theta, Z] = \begin{bmatrix} \theta_1 & -\theta_2 \\ \theta_2 & \theta_1 \end{bmatrix}, \quad [\theta, Z] = \begin{bmatrix} \theta_1 & -\theta_2 & -\theta_3 & \theta_4 \\ \theta_2 & \theta_1 & -\theta_4 & -\theta_3 \\ \theta_3 & -\theta_4 & \theta_1 & -\theta_2 \\ \theta_4 & \theta_3 & \theta_2 & \theta_1 \end{bmatrix}$$

[7] R. Bott and J. Milnor, "On the parallelizability of the spheres," Bulletin of the American Mathematical Society, vol. 64, no. 3.P1, pp. 87 – 89, 1958.

# Topological Properties ($n = 3$)

$S^2$ is not parallelizable (unfortunately)

But the cross product $\times$ exists!

Define a new direction $b \in S^2$, coordinate frame of

$$\begin{bmatrix} \theta & Z \end{bmatrix} = \begin{bmatrix} \theta & b & \theta \times b \end{bmatrix}$$

Support set (with $SO(2)$ symmetry)

$$\Psi_L^3 = \left\{ (\theta, t, b, y) \in \Omega \times S^2 \times B_R^2 \mid \begin{array}{c} \theta t + b y_1 + (\theta \times b) y_2 \in L \\ \theta \cdot b = 0 \} \end{array} \right\}$$

# Topological Properties: Complexity Reduction

Massive savings in computational complexity possible:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| # variables | **4** | **9** | **8** | 35 | 48 | 63 | **16** | 99 |

But 8 variables is still too big for most SOS methods.

# Stokes Constraints

# Stokes Constraints: Volume Approximation

Smooth set $L = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$,

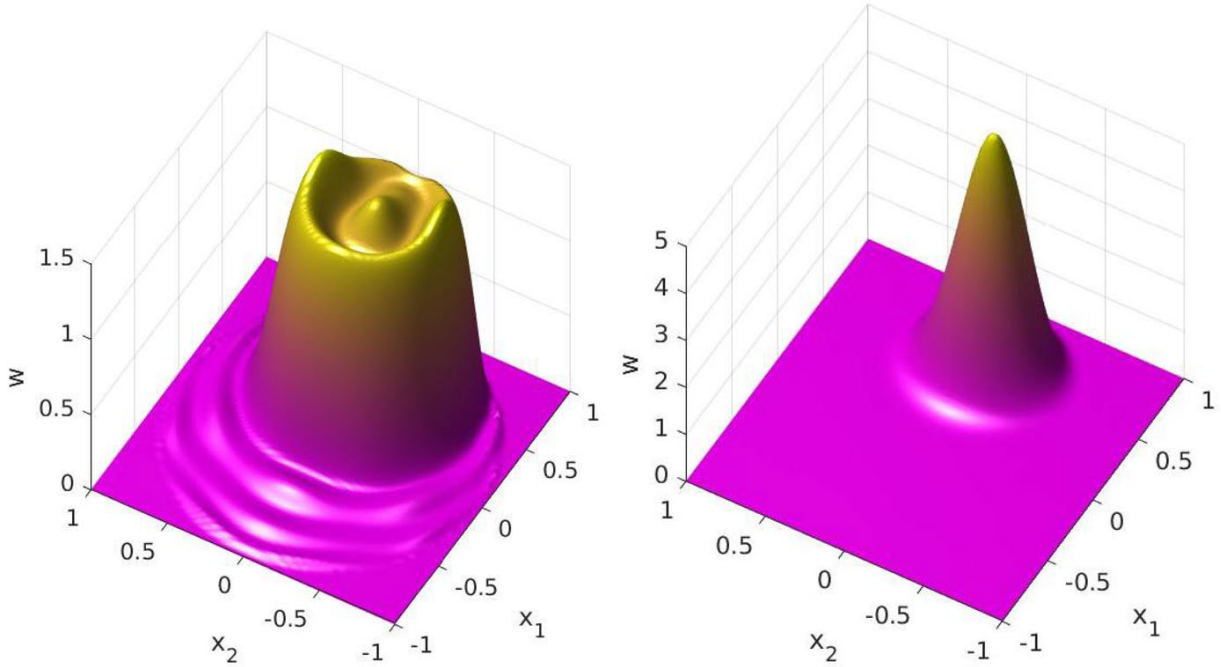
Redundancy: $u(x)g(x)$ integrates to 0 on $\partial L$ (where $g = 0$)

Better accuracy at degree-$k$, avoids discontinuities[8]

$$
\begin{aligned}
V^* = \inf \textstyle\int_X w(x)dx & & & \text{(3a)}\\
w(x) \geq 0 & & \forall x \in X & \text{(3b)}\\
w(x) - 1 - \nabla \cdot u(x) \geq 0 & & \forall x \in L & \text{(3c)}\\
-u(x) \cdot \nabla g(x) \geq 0 & & \forall x \in \partial L & \text{(3d)}\\
w(x) \in C(X),\ u(x) \in [C^1(L)]^n & & & \text{(3e)}
\end{aligned}
$$

[8] M. Tacchi, J. B. Lasserre, and D. Henrion, "Stokes, Gibbs, and Volume Computation of Semi-Algebraic Sets," Discrete & Computational Geometry, vol. 69, no. 1, pp. 260–283, 2023

# Stokes Comparison




Degree $k = 16$ approximation, Left: Indicator, Right: Stokes[9]

| Method | Indicator | Stokes | True |
|---|---|---|---|
| Area | 1.1626 | 0.7870 | $\pi/4 \approx 0.7854$ |

[9] image from "Stokes, Gibbs, and Volume Computation of Semi-Algebraic Sets" (8)

# Slice-Volume Stokes

Group average w.r.t. group $G$: $[\cdot]_G$ (e.g. discrete reflection)

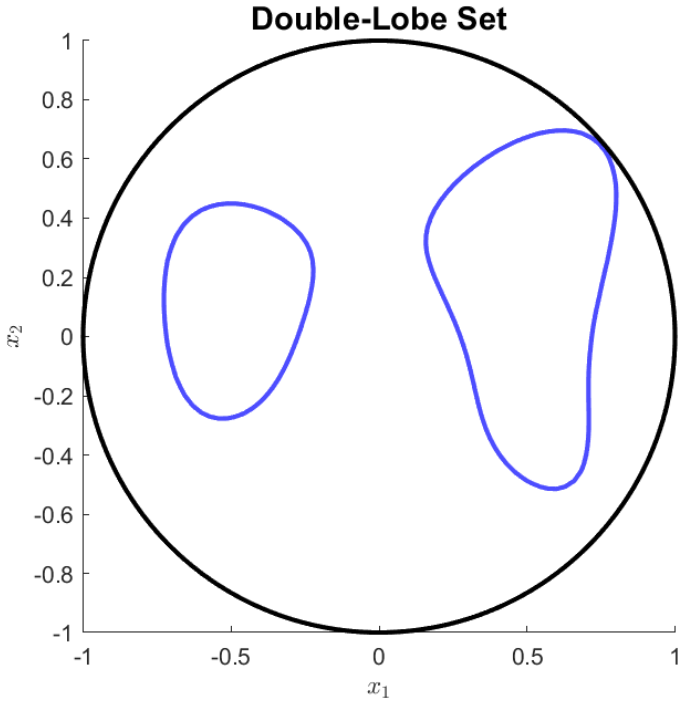
Boundary $\Psi_L^i = \Psi_L^i \cap \{x \mid g_i(\theta t + Zy) = 0\}$

$$
\begin{aligned}
p_s^* = \inf_{\gamma \in \mathbb{R}} \quad & \gamma \\
& \gamma \geq \Lambda_R w(\theta, t, Z) && \forall (\theta, t, Z) \in \Omega_Z \\
& w(\theta, t, Z, y) \geq 1 + [\nabla_y \cdot u(\theta, t, Z, y)]_G && \forall (\theta, t, Z, y) \in \Psi_L \\
& - [u(\theta, t, Z, y) \cdot \nabla_y g_i(\theta t + Zy)]_G \geq 0 && \forall (\theta, t, Z, y) \in \Psi_L^i \\
& w(\theta, t, Z, y) \geq 0 && \forall (\theta, t, Z, y) \in \Psi \\
& w(\theta, t, Z, y) \in C(\Psi)^G \\
& u(\theta, t, Z, y) \in [C^{0,0,0,1}(\Psi_L)]^n.
\end{aligned}
$$

Same slice-volume objective $P^*$, better finite-degree bounds

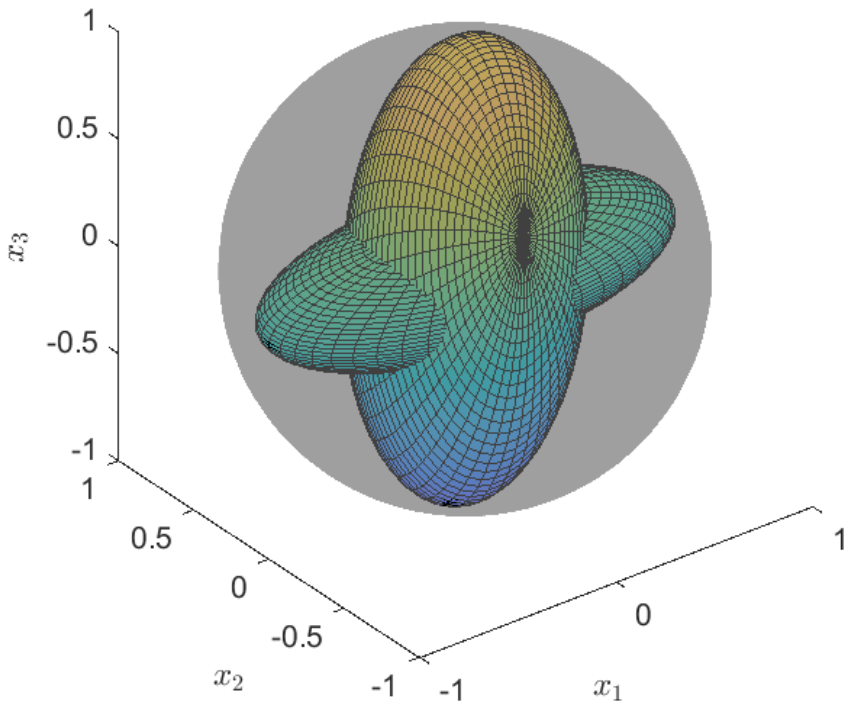
# Examples

# Double-Lobe Set (rotation and translation)




| Order ($k$) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Indicator | 2.0 | 2.0 | 1.9910 | 1.9833 | 1.8608 | 1.8294 |
| Stokes | 2.0 | 2.0 | 1.9799 | 1.6123 | 1.5409 | 1.4728 |

# Union of Ellipses (only translation)




| Order ($k$) | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| Indicator | 3.1029 | 2.9132 | 2.8556 | 2.7739 | 2.6986 | 2.6653 |
| Stokes | 2.8013 | 2.5739 | 2.2680 | 2.1333 | 2.1261 | 2.0814 |

# Take-aways

# Conclusion

Solved the Slice-Volume problem using SOS methods

Reduced complexity (symmetry, algebraic, topological)

Incorporated Stokes constraints for better accuracy

# Future Work

- Local coordinate frames
- Exploit continuous symmetry (SAGBI)
- Faster numerical solutions
- Real rational periods

# Acknowledgements

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# Thank you for your attention!

https://github.com/jarmill/slice_volume

# Slice-Volume Measure Program

$\mu_0$ is a probability distribution over $(\theta, t, Z)$ :

$\mu$ is distributed over $(\theta, t, Z, y)$ in $L$ $(\Psi_L)$

Objective $\langle 1, \mu \rangle$ integrates the slice-volume

$$
\begin{aligned}
m^* = & \sup \langle 1, \mu \rangle \\
& \mu_0 \otimes \lambda_R^{n-1} = \mu + \hat{\mu} \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu_0 \in \mathcal{M}_+(\Omega_Z), \\
& \hat{\mu} \in \mathcal{M}_+(\Psi) \\
& \mu \in \mathcal{M}_+(\Psi_L).
\end{aligned}
$$