Safety Analysis and Control using Measures

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**Motivation**
Quantify the safety of trajectories

Use convex optimization to compute converging distance-lower-bounds

**Peak Estimation**
Utilize theory of peak estimation

Find bounds on state function \( p(x) \)

\[
P^* = \min_{t, x_0 \in X_0} p(x(t | x_0))
\]

\[
\dot{x}(t) = f(t, x(t)) \quad t \in [0, T]
\]

Finite dimensional but nonconvex

**Occupation Measures**
Returns the time trajectories spend in each set (given initial distribution)

Convex but infinite-dimensional (LP)

**Distance Estimation**
Distance \( c(x, y) \) (e.g. Euclidean \( \|x - y\|_2 \))

Unsafe-set \( c(x; X_u) = \min_{y \in X_u} c(x, y) \)

Peak estimation with \( p(x) = c(x; X_u) \)

Variable groups \((t, x)\) and \((x, y)\)

Approx. recovery: rank-1 solutions

Separability reduces SDP complexity: \( c(x, y) = \sum_{k=1}^n c_k(x_k, y_k) \)

Safety of Points

Safety of Shapes (rotating square)

**Hybrid Dynamics**
Transitions between locations

Guards and resets (e.g. contact)

Uncontrolled to infinity

Controlled to gray region

Further details and papers

**Uncertainty**
Compactly supported uncertainty

Time-dependent (bounded noise) or time-independent (parametric)

Exploit polytopic and switching structure

**Stochastic Dynamics**
Probabilistic bounds on peak/distance

Use upper-bounds of Value-at-Risk

Dotted: 50% bound, Solid: 85% bound

**Time Delay**
Discrete time delay \( \tau \), history \( x_h \)

\[
\dot{x}(t) = f(t, x(t), x(t - \tau)) \quad t \in [0, T]
\]

\[
x(s) = x_h(s) \quad s \in [-\tau, 0]
\]

Gray: history, cyan: system trajectory