Facial Input Decompositions for Robust Peak Estimation under Polyhedral Uncertainty

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Main Ideas

Peak estimation problems posed under polyhedral uncertainty

Use polytopic structure to simplify Lie constraints

Apply to L_∞ bounded noise setting

Peak Estimation Background

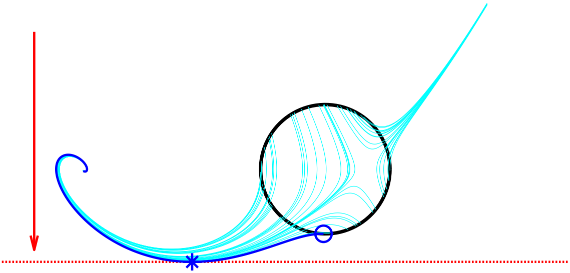
Peak Estimation Problem

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t,\ x_0 \in X_0} p(x(t))$$

$$\dot{x}(t) = f(t, x(t)) \qquad t \in [0, T]$$

$$x(0) = x_0 \in X_0$$



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Peak Function Program

Infinite dimensional linear program (Fantuzzi, Goluskin, 2020)

Uses auxiliary function $v(t, x)$

$$d^* = \min_{\gamma \in \mathbb{R}} \quad \gamma \tag{1}$$

$$\gamma \geq v(0, x) \qquad \forall x \in X_0 \tag{2}$$

$$\mathcal{L}_f v(t, x) \leq 0 \qquad \forall (t, x) \in [0, T] \times X \tag{3}$$

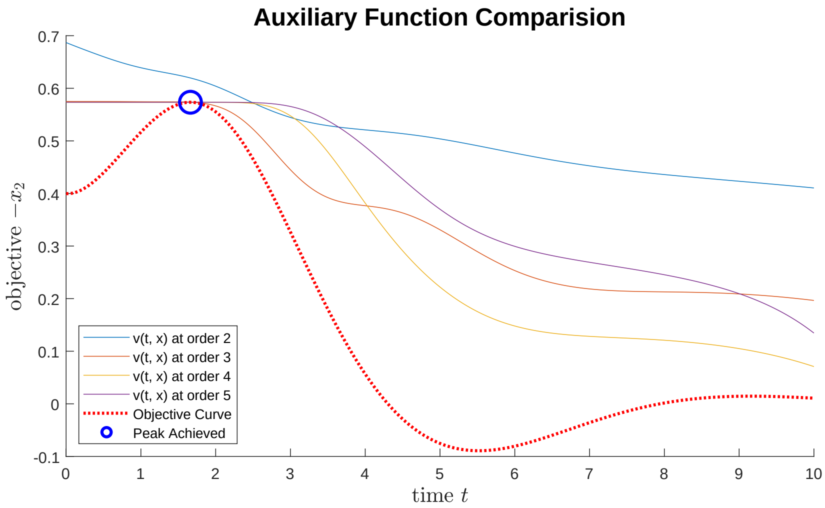
$$v(t, x) \geq p(x) \qquad \forall (t, x) \in [0, T] \times X \tag{4}$$

$$v \in C^1([0, T] \times X) \tag{5}$$

Lie Derivative $\mathcal{L}_f v(t, x) = \partial_t v + f(t, x) \cdot \nabla_x v$

$P^* = d^*$ holds if $[0, T] \times X$ is compact, f Lipschitz

Auxiliary Evaluation along Optimal Trajectory



Optimal $v(t, x)$ should be constant until peak is achieved

Peak Estimation with Uncertainty

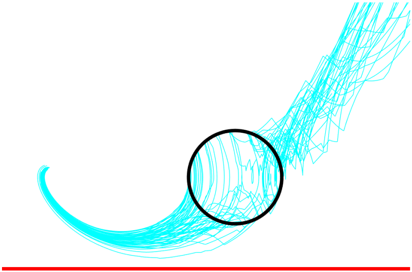
Dynamics $\dot{x} = f(t, x(t), w(t))$

Uncertain process $w(t) \in W, \ \forall t \in [0, T]$

Time-dependent w

$$\mathcal{L}_{f(t,x,w)} v(t,x) \leq 0 \qquad \forall (t, x, w) \in \forall [0, T] \times X \times W$$

System with Uncertainty Example



$$\dot{x}(t) = [x_2(t), -x_1 w(t) - x_2(t) + x_1(t)^3/3]$$

$$w(t) \in [0.5, 1.5]$$

Assumptions

Set $[0, T] \times X$ is compact

Uncertainty W is a compact polytope $\{w \mid Aw \leq b\}$

Nonempty interior: $\exists w \in \mathbb{R}^L \mid Aw < b$

Dynamics $f(t, x, w)$ are Lipschitz

Input-affine $f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x)$

Constraint Decomposition

Feasibility Pair

Polyhedral-constrained input $w(t) \in W$

Problem is Feasible

$$\mathcal{L}_{f(t,x,w)} v(t,x) \leq 0 \qquad \forall (t,x,w) \in [0,T] \times X \times W$$

Problem is Infeasible

$$\mathcal{L}_{f(t,x,w)} v(t,x) > 0 \qquad \exists (t,x,w) \in [0,T] \times X \times W$$

Pair of Strong Alternatives

Theorem of Alternatives

Lagrange dual function g

$$g(\zeta) = \sup_{w \in \mathbb{R}^L} \ S(w;\zeta) = \sup_{w \in \mathbb{R}^L} \mathcal{L}_f v + \sum_{k=1}^{m} \zeta_k(t,x)(b_k - A_k w)$$

Certificate ζ that $\mathcal{L}_f v > 0$ is empty:

$$\forall (t,x) \in [0,T] \times X : \qquad g(\zeta) \leq 0, \quad \zeta(t,x) \geq 0$$

Weak alternative $g(\zeta) \leq 0$ also strong:

- w-affine functions $\mathcal{L}_f v$ and $b - Aw$ are w-concave
- Exists a point $w : Aw < b$ (Slater)

Lie Polytopic Decomposition

Original

$$\mathcal{L}_f v(t,x) \leq 0 \qquad \forall (t,x,w) \in [0,T] \times X \times W$$

Decomposed

$$\begin{aligned} &\mathcal{L}_{f_0} v(t,x) + b^T \zeta(t,x) \leq 0 && \forall (t,x) \in [0,T] \times X \\ &(A^T)_\ell \zeta(t,x) = f_\ell \cdot \nabla_x v(t,x) && \forall \ell = 1,\ldots,L \\ &\zeta_k(t,x) \in C_+([0,T] \times X) && \forall k = 1,\ldots,m \end{aligned}$$

Strong equivalence (given affine structure in w)

Peak Estimation (revisited)

Peak Original Program

Include time-varying uncertainty $w(t) \in W$

$$
\begin{aligned}
d^* = \min_{\gamma \in \mathbb{R}} \quad & \gamma \\
& \gamma \geq v(0, x) && \forall x \in X_0 \\
& \mathcal{L}_f v(t, x) \leq 0 && \forall (t, x, w) \in [0, T] \times X \times W \\
& v(t, x) \geq p(x) && \forall (t, x) \in [0, T] \times X \\
& v \in C^1([0, T] \times X)
\end{aligned}
$$

Peak Decomposed Program

Only the Lie Derivative constraint changes

$$
\begin{aligned}
d^* = \min_{\gamma \in \mathbb{R}} & \ \gamma \\
& \gamma \geq v(0, x) & & \forall x \in X_0 \\
& \mathcal{L}_{f_0} v(t, x) + b^T \zeta(t, x) \leq 0 & & \forall (t, x) \in [0, T] \times X \\
& (A^T)_\ell \zeta(t, x) = (f_\ell \cdot \nabla_x) v(t, x) & & \forall \ell = 1, \ldots, L \\
& v(t, x) \geq p(x) & & \forall (t, x) \in [0, T] \times X \\
& v(t, x) \in C^1([0, T] \times X) & & \\
& \zeta_k(t, x) \in C_+([0, T] \times X) & & \forall k = 1, \ldots, m
\end{aligned}
$$

Data Driven Setting

Noise Model

Ground truth $\dot{x} = F(t, x)$

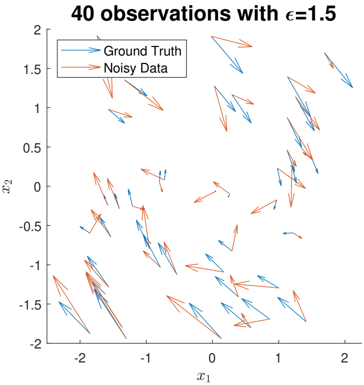
Corrupted observations of system F in $t \in [0, T]$

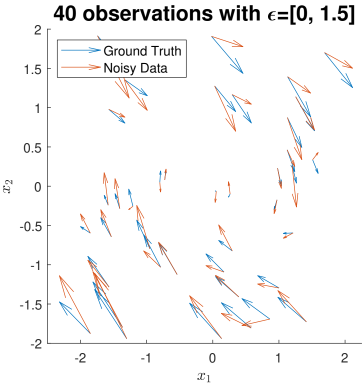
$$(t_j, x_j, \dot{x}_j) \qquad \forall j = 1, \ldots, N_s$$

Assumption of L_∞ bounded noise

$$\|F(t_j, x_j) - \dot{x}_j\|_\infty \leq \epsilon \qquad \forall j = 1, \ldots, N_s$$

Sampling: Flow System





$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Dynamics Model

Parameterize unknown F by functions in dictionary

$$\dot{x}(t) = f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x)$$

Affine in uncertainties w

Bounded noise constraint ϵ

$$\begin{aligned} \|F(t_j, x_j) - \dot{x}_j\|_\infty &\leq \epsilon & \forall j = 1, \ldots, N_s \\ \|f(t_j, x_j, w) - \dot{x}_j\|_\infty &\leq \epsilon & \forall j = 1, \ldots, N_s \end{aligned}$$

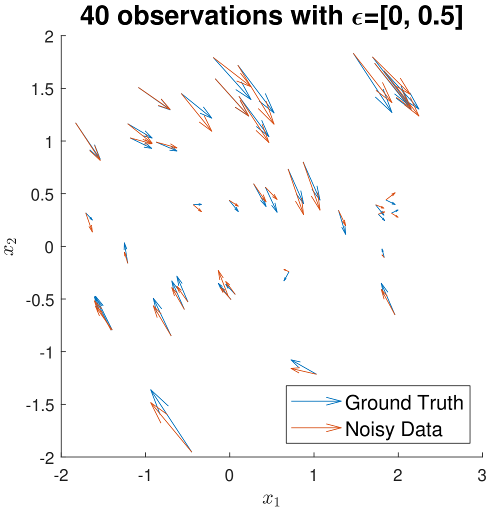
Noise Constraints

2 linear constraints for each coordinate i, sample j

$$-\epsilon \leq f_0(t_j, x_j)_i + \sum_{\ell=1}^{L} w_\ell f_\ell(t_j, x_j)_i - (\dot{x}_j)_i \leq \epsilon$$

Polytopic region $W = \{w \in \mathbb{R}^L \mid Aw \leq b\}$ with $b \in \mathbb{R}^{2N_x N_s}$

Peak Estimation Example (Flow)



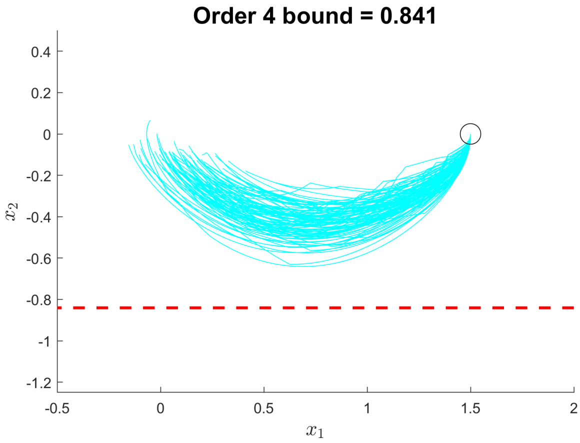
$$\dot{x} = [x_2, \; -x_1 - x_2 + x_1^3/3], \;\; T = 5$$

Peak Estimation Example (Flow)

$$\dot{x} = [x_2,\ -wx_1 - x_2 + x_1^3/3]$$

$$L = 1,\ m = 80 \text{ (2 nonredundant)}$$

Peak Estimation Example (Flow)



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

$$L = 10,\ \ m = 80\ (33\ \text{nonredundant})$$

Computational Considerations (Flow)

Size of largest PSD matrix in SDP (without symmetries):

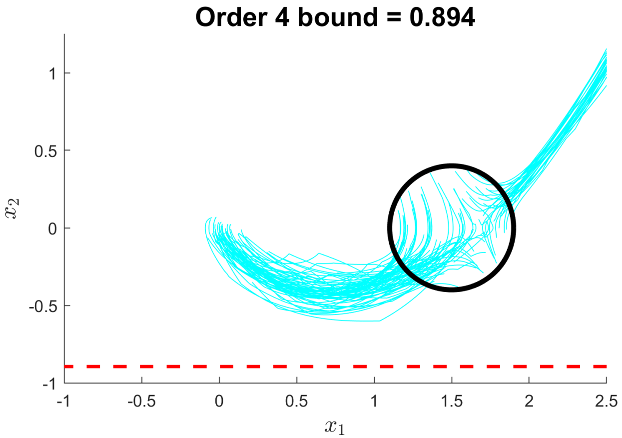
$$\text{Original} \quad \binom{1+N_x+L+d+\lceil \deg(f)/2\rceil-1}{1+N_x+L} = \binom{18}{13} = 8568$$

$$\text{Decomposed} \quad \binom{1+N_x+d+\lceil \deg(f)/2\rceil-1}{1+N_x} = \binom{8}{3} = 56$$

Order $d = 4$, $L = 10$, $N_x = 2$

W has 33 faces, 7534 vertices

Peak Estimation Example (Flow)



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$

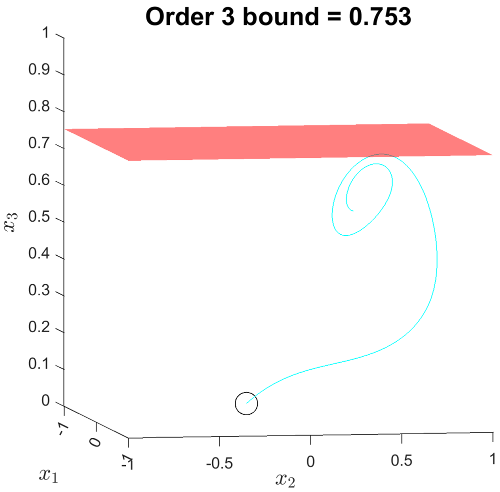
Peak Estimation Example (Twist)

Dynamics model:

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)$$

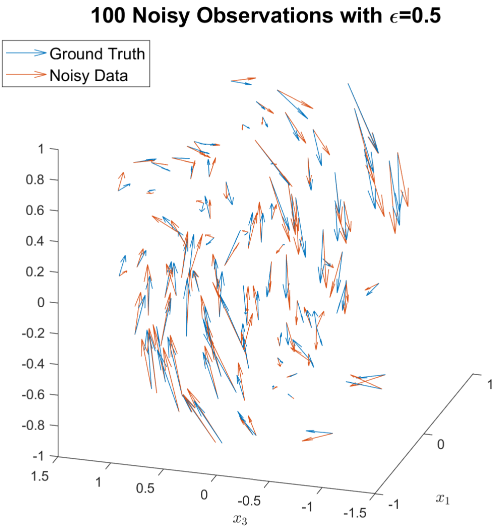
$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$



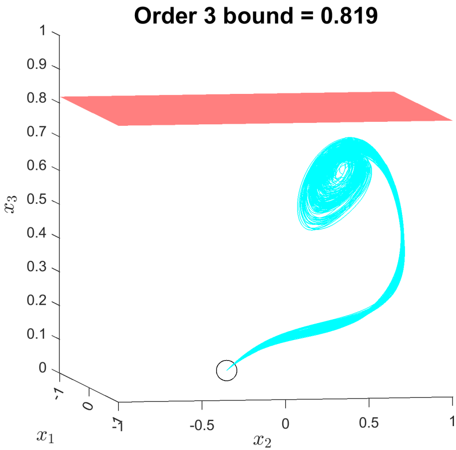
$$X_0 = [-1, 0, 0], \quad T = 8$$

Peak Estimation Example (Twist)



$m = 2N_sN_x = 600$ constraints

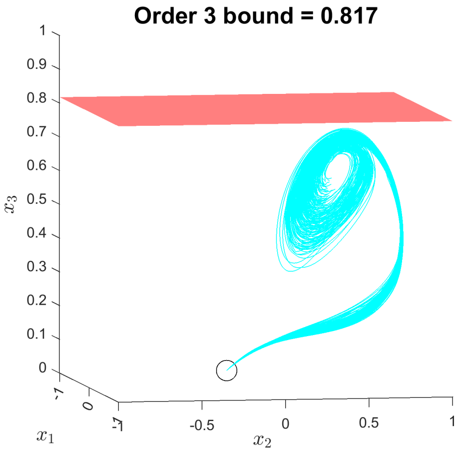
Peak Estimation Example (Twist)



Unknown A, Known B

$$L = 9, \ m = 600 \ (34 \text{ nonredundant})$$

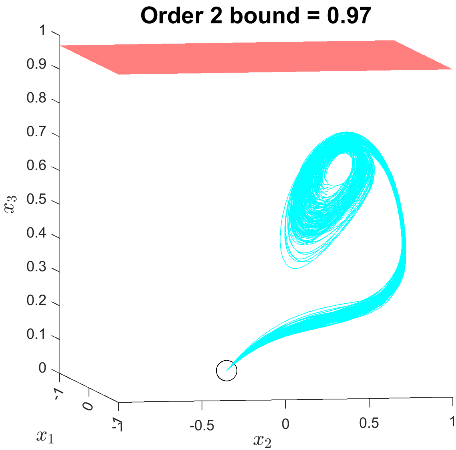
Peak Estimation Example (Twist)



Known A, Unknown B

$$L = 9, \;\; m = 600 \text{ (30 nonredundant)}$$

Peak Estimation Example (Twist)



Unknown A, Unknown B

$L = 18, \; m = 600$ (70 nonredundant)

Take-aways

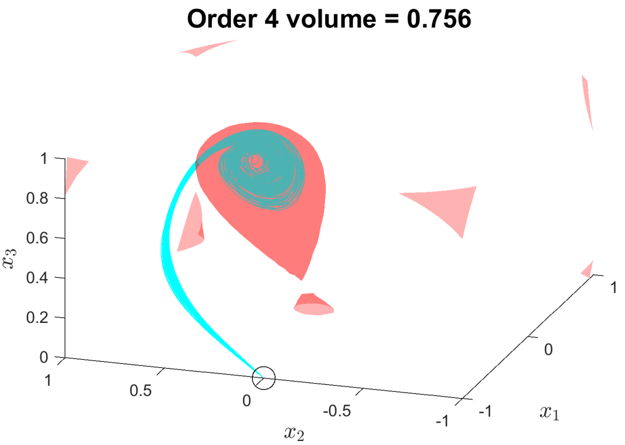
Conclusion

Tractable peak estimation problems (after preprocessing)

More SOS constraints in fewer variables

Data-driven estimates given L_∞-bounded noise

Reachable Set Estimation Example (Twist)



Unknown A, Known B

$L = 9, \; m = 600$ (34 nonredundant)

Future Work

- Streaming data and warm starts
- Maximum positively invariant sets
- Hybrid systems
- Compatibility with structure (e.g. sparsity)

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Thank you for your attention

arxiv:2112.14838

github.com/jarmill/data_driven_occ

Extra Material

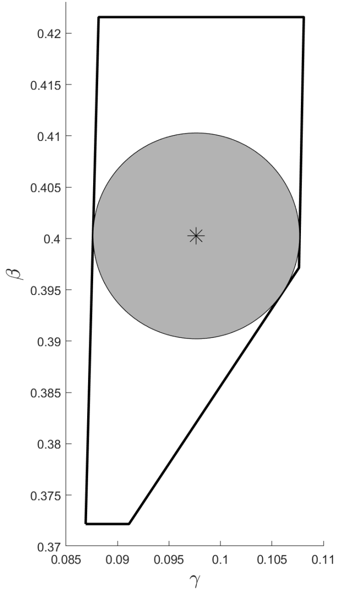
Preprocessing: Centering

Chebyshev center c: center of sphere with largest radius in W

Find through linear programming

$$\begin{aligned} &\max r \\ &A_k c + r\|A_k\|_2 \leq b_k \qquad \forall k \\ &r \geq 0, c \in \mathbb{R}^L \end{aligned}$$

Shifted dynamics $f_0 \leftarrow f_0 + \sum_{\ell=1}^{L} c_\ell f_\ell$



Preprocessing: Redundancy

Majority of $m = 2N_x N_s$ constraints are often redundant

Convex hull of dual polytope:
Time: $\Omega(m \log m + m^{\lfloor L/2 \rfloor})$

Linear program per constraint:
Time: $m \times \tilde{O}(mL + L^3)$
(Jan van den Brand *et. al.* 2020)