Data-Driven Control of Positive Linear Systems using Linear Programming

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IEEE CDC: WeB17.1

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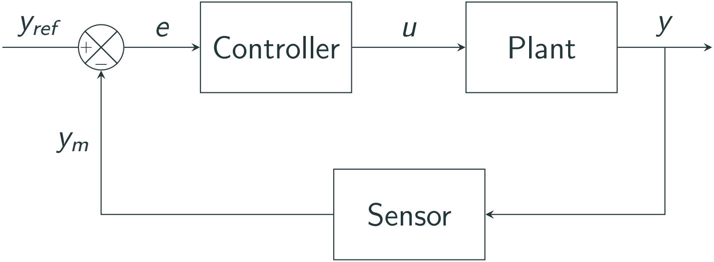
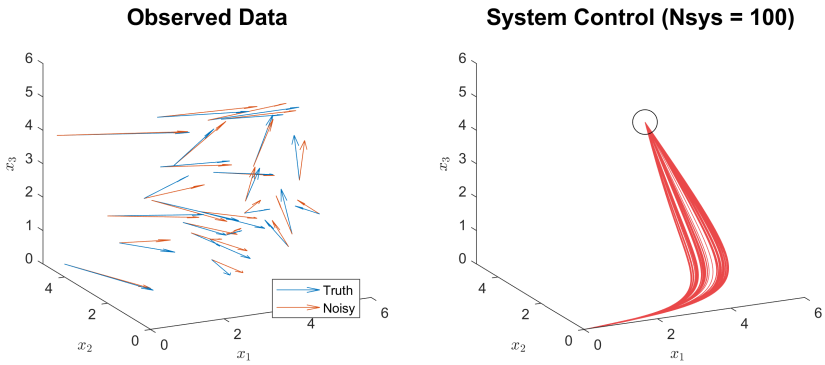
What is Data-Driven Control?

Design a controller for an unknown plant



Control system directly from data, no sysid required

Sneak Preview of Positive-Stabilization



Single controller stabilizes all data-consistent plants

Algorithms for Data-Driven Control

Desired model: Virtual Reference Feedback Tuning

Set-Membership (this talk)

- (Data-consistent plants) $\subseteq$ (K-Stabilized plants)
- Certificates: Farkas, Interval, S-Lemma, SOS

Behavioral

- Parameterize and pick out best system trajectory (MPC)
- Willem's Fundamental Lemma (DeePC)

Learning: Koopman, Gaussian Processes, Regression

Flow of Presentation

Describe positive systems

Pose data-driven positive stabilization problem

Demonstrate on example systems

Positive Systems

What are Positive Systems?

Dynamics $\delta x(t) = f(x(t), u(t))$ such that (full-state):

$$x(0), u(t) \geq 0 \implies x(t) \geq 0 \tag{1}$$

Internal/External positivity under input-output

Applications in:

- Compartmental Models
- Chemical Reaction Networks
- Economics
- Networked Control

Stability of Positive Systems

System $\delta x = Ax$ is positive-stable if:

Continuous-time:	A is Hurwitz and Metzler
Discrete-time:	A is Schur and Nonnegative

$\forall t : x(t) \geq 0$ and $\lim_{t \to 0} x(t) = \mathbf{0}$

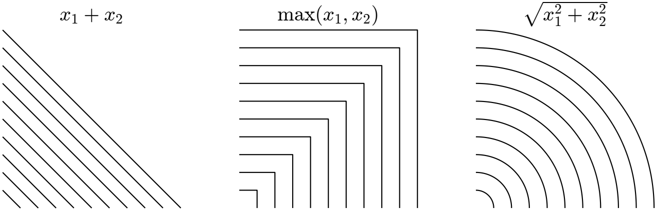
Controlled system: B must be nonnegative

(Metzler: nonnegative off-diagonals)

Copositive Lyapunov Functions

Functions $V(x)$ satisfying Lyapunov inequalities over $\mathbb{R}^+_{\geq 0}$:

Linear:	$V(x) = \sum_{i=1}^n a_i x_i,$	$a \in \mathbb{R}^n_{>0}$
Dual Linear:	$V(x) = \max_i x_i / v_i$	$v \in \mathbb{R}^n_{>0}$
Quadratic:	$V(x) = x^T M x$	$M \in \mathrm{COP}^n$



Positive Stabilization

Positive-stabilize $\delta x = Ax + Bu$ using $u = Kx$ with $K \in \mathbb{R}^{m \times n}$:

Copositive CLF $\max x./v$ with $X = \text{diag}(v),\ \ Y \in \mathbb{R}^{m \times n}$

Continuous-time system:

$$-(AX + BY)\mathbf{1}_n \in \mathbb{R}^n_{>0} \qquad AX + BY \text{ is Metzler} \tag{2a}$$

Discrete-time system:

$$v - (AX + BY)\mathbf{1}_n \in \mathbb{R}^n_{>0} \qquad AX + BY \in \mathbb{R}^{n \times n}_{\geq 0} \tag{2b}$$

If system is feasible, then recover controller by $K = YX^{-1}$

Data-Driven Positive Stabilization

Consistency Set

Collect data $\mathcal{D}$ with

$$\begin{array}{rcllll} \mathbf{X} & := & [x(0) & x(1) & \dots & x(T-1)] \\ \mathbf{U} & := & [u(0) & u(1) & \dots & u(T-1)] \\ \mathbf{X}_\delta & := & [\delta x(0) & \delta x(1) & \dots & \delta x(T-1)] \end{array} \tag{3}$$

Discrepancy matrix $\mathbf{W}$ from observation

$$\mathbf{W} = \mathbf{X}_\delta - (A\mathbf{X} + B\mathbf{U}) \tag{4}$$

Assumption on $\mathbf{W}$: $\forall t : \ \|w(t)\|_\infty \leq \epsilon$

Sets for Positive-Stabilization

Form polytopic sets of plants (A, B)

P_1 : Set of plants consistent with data $\mathcal{D}$

$$P_1 : \ \{(A, B) \mid \|\delta x(t) - (Ax(t) - Bu(t))\|_\infty \leq \epsilon\} \tag{5}$$

P_2 : Set of plants positive-stabilized by $K = YX^{-1}$ $(\eta > 0)$

$$-(AX + BY)\mathbf{1}_n - \eta\mathbf{1}_n \in \mathbb{R}_{\geq 0} \quad \text{(continuous-time)} \tag{6a}$$

$$v - (AX + BY)\mathbf{1}_n - \eta\mathbf{1}_n \in \mathbb{R}_{\geq 0} \quad \text{(discrete-time).} \tag{6b}$$

Set-Membership Containment

Positive-stabilization occurs if $P_1 \subseteq P_2$, need to certify

Extended Farkas Lemma (Hennet 1989)

For sets $P_1 = \{x \mid G_1 x \leq h_1\}$ and $P_2 = \{x \mid G_2 x \leq h_2\}$, the relation $P_1 \subseteq P_2$ iff $\exists Z \in \mathbb{R}_{\geq 0}^{c_2 \times c_1}$ with:

$$ZG_1 = G_2, \qquad Zh_1 \leq h_2. \tag{7}$$

Use Extended Farkas Lemma to prove containment

Set-Membership Positive Stabilization

Stabilizing set P_2 depends on v, Y

Choose v, Y to attempt satisfaction of:

$$\begin{aligned}
\underset{v,Y,Z}{\text{find}} \quad & ZG_1 = G_2(v, Y), \qquad Zh_1 \leq h_2(v, Y) && \text{(8a)} \\
& v - \eta \mathbf{1}_n \in \mathbb{R}^n_{\geq 0} && \text{(8b)} \\
& Y \in \mathcal{S} && \text{(8c)} \\
& Z \in \mathbb{R}^{q \times 2nT}_{\geq 0}, && \text{(8d)}
\end{aligned}$$

If successful, recover controller with $K = Y\text{diag}(1./v)$

Computational Complexity

Solve a single LP to find v, Y such that $P_1 \subseteq P_2$

	# Ineq.	# Eq.
v	n	0
Y	$\leq mn$	0
Z	$2nTq$	0
Farkas	q	$qn(n+m)$

More efficient than existing LMI based methods for data-driven positive-stabilization

Switching Behavior

Adapt literature on switched positive systems towards data-driven setting

- Arbitrary switching permitted
- Linear-Parameter Varying dynamics
- Discrete-time switching on transition graph

Cannot yet handle continuous-time dwell-time constraints

Peak-to-Peak Gain

Peak-to-Peak Gain Setup

Disturbance term ξ, controlled output z

$$\delta x(t) = Ax(t) + Bu(t) + E\xi(t) \tag{9a}$$

$$z(t) = Cx(t) + Du(t) + F\xi(t). \tag{9b}$$

Assume C, D, E, F are known nonnegative matrices

Minimize peak-to-peak (p2p) gain $\xi \to z$

Peak-to-Peak Gain Program

Continuous-time p2p gain $\leq \gamma$ if:

$$
\begin{aligned}
&-(AX + BY)\mathbf{1}_n - E\mathbf{1}_e \in \mathbb{R}^n_{>0} && \text{(10a)}\\
&\gamma \mathbf{1}_q - (CX + DY)\mathbf{1}_n - F\mathbf{1}_e \in \mathbb{R}^q_{>0} && \text{(10b)}\\
&CX + DY \in \mathbb{R}^{q \times n}_{\geq 0} && \text{(10c)}\\
&AX + BY \text{ is Metzler} && \text{(10d)}
\end{aligned}
$$

Infimize γ for suboptimal control

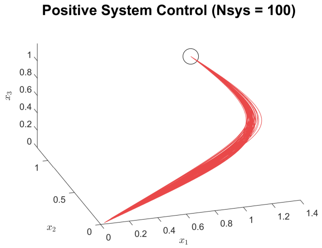
Use (10a) and (10a) to form P_2 given $\mathcal{D}$

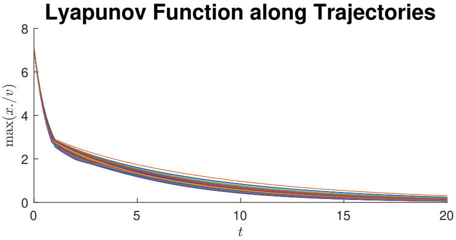
Examples

Example 1: Continuous-Time System

Continuous-time system with $T = 5$ samples, $\epsilon = 0.1$

$$A = \begin{bmatrix} -0.55 & 0.3 & 0.65 \\ 0.06 & -1.35 & 0.25 \\ 0.1 & 0.15 & 0.4 \end{bmatrix} \qquad B = \begin{bmatrix} 0.18 & 0.08 \\ 0.47 & 0.25 \\ 0.07 & 0.95 \end{bmatrix}$$





Example 2: Discrete-Time System

Plant with $n = 5$ states and $m = 3$ inputs

Sign pattern ($\oplus$ nonneg, $\ominus$ nonpos, $\odot$ zero, $\circledast$ arbitrary)

$$\mathcal{S} = \begin{bmatrix} \odot & \odot & \odot & \odot & \ominus \\ \odot & \odot & \circledast & \odot & \oplus \\ \odot & \odot & \odot & \circledast & \circledast \end{bmatrix}$$

Derived certificate and controller

$$v = \begin{bmatrix} 0.2147 & 0.1259 & 0.2448 & 0.2516 & 0.1630 \end{bmatrix}^T$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.6853 \\ 0 & 0 & -0.3206 & 0 & 0.1206 \\ 0 & 0 & 0 & -0.5604 & -0.3317 \end{bmatrix}$$

Example 3: Peak-to-Peak Gain

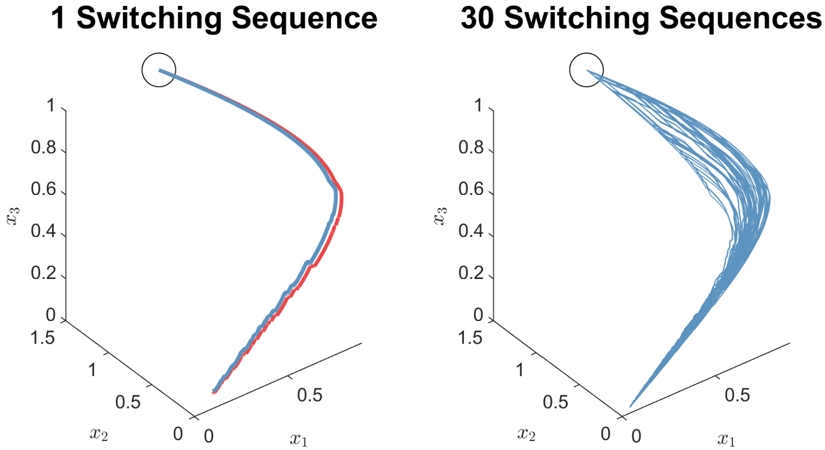
Worst-case Peak-to-Peak gain falls as T rises

T	20	30	50	80	120
No Prior	6.4823	5.0719	4.5292	4.0659	4.0029
A Metzler	6.4539	5.0182	4.4967	4.0619	4.0028

Can prior knowledge to (A, B) by augmenting P_1

Example 4: Stabilization under Arbitrary Switching

Both linear subsystems are positive-stabilized by same K



Take-aways

Conclusion

Positive-stabilization of unknown systems

Requires bounds on process noise

Simple linear programming problems

Nonconservative (up to common dual linear copositive CLFs)

Acknowledgements

Roy Smith, Automatic Control Lab (IfA)

POP group at LAAS-CNRS

NCCR Automation

Air Force Office for Scientific Research

National Science Foundation

Thanks!

Questions?