

Data-Driven Stabilizing and Robust Control of Discrete-Time Linear Systems with Error in Variables

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Abstract

This work presents a sum-of-squares (SOS) based framework to perform data-driven stabilization and robust control tasks on discrete-time linear systems where the full-state observations are corrupted by L-infinity bounded measurement noise (error in variable setting). Certificates of state-feedback superstability or quadratic stability of all plants in a consistency set are provided by solving a feasibility program formed by polynomial nonnegativity constraints. Under mild compactness and data-collection assumptions, SOS tightenings in rising degree will converge to recover the true superstabilizing controller, with slight conservatism introduced for quadratic stabilizability. The performance of this SOS method is improved through the application of a theorem of alternatives while retaining tightness, in which the unknown noise variables are eliminated from the consistency set description. This SOS feasibility method is extended to provide worst-case-optimal robust controllers under H2 control costs. The consistency set description may be broadened to include cases where the data and process are affected by a combination of L-infinity bounded measurement, process, and input noise. Further generalizations include varying noise sets, non-uniform sampling, and switched systems stabilization.

1 Introduction

Data-driven control is a group of methods that sidestep a system-identification step in order to design controllers that regulate all possible plants that are consistent with observed data. The dynamical model considered in this paper is a discrete-time linear system with states $x_t \in \mathbb{R}^n$ and inputs $u_t \in \mathbb{R}^m$, for which measured data up to a finite time horizon of T is available as $\mathcal{D} = \{\hat{u}_t, \hat{x}_t\}_{t=1}^T$. The system includes L_∞ -bounded full-state measurement noise Δx_t , input noise Δu_t , and process noise w_t to form the model,

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{1a}$$

$$\hat{x}_t = x_t + \Delta x_t, \quad \hat{u}_t = u_t + \Delta u_t, \tag{1b}$$

It is desired to find a constant matrix K generating state-feedback law $u = Kx$ to perform stabilization or robust optimal control of all plants that could have generated the data in \mathcal{D} . Expressing the model (1) purely in terms of observations \mathcal{D} and noise processes $(\Delta x, \Delta u, w)$,

$$\hat{x}_{t+1} - \Delta x_{t+1} = A(\hat{x}_t - \Delta x_t) + B(u_t - \Delta u_t) - w_t. \tag{2}$$

Equation (2) involves multiplications $(A\Delta x_t, B\Delta u_t)$ between unknown variables, which significantly increases the complexity of finding controllers K (these bilinearities typically yield NP-hard problems). This work formulates control of all consistent plants as a Polynomial Optimization Problem (POP), which is approximated

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by a converging sequence of Semidefinite Programs (SDPs) through Sum of Squares (SOS) methods [1]. A theorem of alternatives based on robust SDPs is used to reduce the complexity of the generated SOS programs by eliminating the noise variables $(\Delta x, \Delta u, w)$.

Most prior work on data-driven control involves process noise w alone, and sets $\Delta x, \Delta u = 0$. One such method includes the work in [2], which utilizes Willem’s fundamental lemma [3] to generate control policies based solely on the input-state-output data \mathcal{D} . The method in [2] presents a set of data-driven programs including stabilization and LQR control. Regularization methods have been applied to use Willem’s fundamental lemma in the robust setting, but bilinear dependence and fragility degrades performance in the derived controllers [2, 4, 5].

Other SDP-based methods for nonconservative control under L_2 -bounded process noise (with $\Delta x, \Delta u = 0$) includes an S-Lemma [6], Petersen’s Lemma [7], and through updating uncertainties [8]. The work in [9] defines a notion of ‘data informativity,’ demonstrating that the assumption required for data-driven stabilization task is less restrictive than what is needed to perform system identification.

The L_∞ noise bound arises from error propagation of finite-difference approximations when computing derivatives and sampling. Another advantage of L_∞ noise as compared to L_2 noise is that multiple datasets \mathcal{D} with differing time horizons can be concatenated without scaling or shifting the noise effects. The work in [8] briefly mentions adaptation for the L_∞ -bounded process noise case while the computational complexity of L_∞ -bounded stabilization increases in an exponential manner with the number of measurements in \mathcal{D} (T). A SOS-based approach addressing the same problem can be found in [10] which demands less complexity. Another way to reduce the complexity is to use the conservative notions of superstability [11]. Further work on L_∞ -bounded process noise for superstabilization may be found in [12–14].

The case where measurement noise $\Delta x \neq 0$ is present is also called the Error in Variables (EIV) setting. Prior work for EIV has mostly concentrated on observation and system identification [15–18]. It is worth noting that in the L_∞ -bounded setting, the set of plants (A, B) consistent with pure process noise w forms a polytope, while the plants consistent with EIV are generically contained in a non-convex region [19]. To the best of our knowledge, this work is the first to deal with EIV stabilization and robust control where the only conservatism is in the choice of superstability or quadratic stability.

The contributions of this paper are,

- Formulation of stabilization under EIV as a polynomial optimization problem
- Application of SOS methods to recover (super and quadratic) constant state-feedback stabilizing controllers of all consistent plants with recorded data
- Simplification of SOS programs by using a Theorem of Alternatives to eliminate affine-dependent noise variables
- Analysis of computational complexity of SOS programs
- Worst-case optimal H_2 control
- Proofs of continuity, polynomial approximability, and convergence

This paper is laid out as follows: Section 2 introduces preliminaries such as acronym definitions, notation, super and quadratic stability of classes of linear systems, and SOS methods. Section 3 creates a Basic Semi-algebraic (BSA) description of the consistency set of plants compatible with measurement-noise-corrupted data, and formulates SOS algorithms to recover (super or quadratic) stabilizing controllers. Section 4 reduces the computational complexity of these SOS programs by eliminating the affine-dependent measurement noise variables through a Theorem of Alternatives. Section 5 applies the Full and Alternative methods to synthesize worst-case-optimal controllers for systems under measurement noise. Section 6 quantifies this reduction in computational complexity by analyzing the size and multiplicities of Positive Semidefinite (PSD) matrices involved in these SOS methods. Section 7 extends the previously presents SOS formulations to problems with measurement, input, and process noise. Section 8 demonstrates the SOS stabilizing algorithms on a set of examples. Section 9 extends the SOS framework to cases including varying noise sets, non-uniform sampling times, and switched systems stabilization (with a known switching sequence). Section 10 concludes the paper. Appendix A proves that multiplier functions for the Alternatives program may be chosen to be continuous. Appendix B builds on this result and proves that the multiplier functions may also be chosen to be symmetric-matrix-valued polynomials.

2 Preliminaries

2.1 Acronyms/Initialisms

BSA	Basic Semialgebraic
CLF	Control Lyapunov Function
EIV	Error in Variables
LMI	Linear Matrix Inequality
LP	Linear Program
PMI	Polynomial Matrix Inequality
POP	Polynomial Optimization Problem
PSD	Positive Semidefinite
PD	Positive Definite
SDP	Semidefinite Program
SOS	Sum of Squares
WSOS	Weighted Sum of Squares

2.2 Notation

The set of real numbers is \mathbb{R} , its n -dimensional vector space is \mathbb{R}^n , and its n -dimensional nonnegative real orthant is \mathbb{R}_+^n . The set of natural numbers is \mathbb{N} , and the subset of natural numbers between 1 and N is $1..N$.

The set of $m \times n$ matrices with real entries is $\mathbb{R}^{m \times n}$. The transpose of a matrix Q is Q^T , and the subset of $n \times n$ symmetric matrices satisfying $Q^T = Q$ is \mathbb{S}^n . The square Identity matrix is $I_n \in \mathbb{S}^n$. The rectangular identity matrix $I_{n \times m}$ matrix whose main diagonal has values of 1 with all other entries equal to zero (consistent with MATLAB's `eye(n,m)` function). The inverse of a matrix $Q \in \mathbb{R}^{n \times n}$ is Q^{-1} , and the inverse of its matrix transpose is Q^{-T} . The trace of a matrix Q is $\text{Tr}(Q)$. The Kronecker product of two matrices A and B is $A \otimes B$. The set of real symmetric PSD matrices \mathbb{S}_+^n have all nonnegative eigenvalues ($Q \succeq 0$), and its subset of Positive Definite (PD) matrices \mathbb{S}_{++}^n have all positive eigenvalues ($Q \succ 0$). The L_∞ operator norm of a matrix $M \in \mathbb{R}^{m \times n}$ is $\|M\|_\infty = \max_i \sum_{j=1}^n |M_{ij}|$. The asterisk operator $*$ may be used to fill in transposed entries of a symmetric matrix. The minimum and maximum eigenvalues of a matrix $Q \in \mathbb{S}^n$ are $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ respectively.

The set of polynomials in variable x with real coefficients is $\mathbb{R}[x]$. The degree of a polynomial $p(x) \in \mathbb{R}[x]$ is $\deg p$. The set of polynomials with degree at most d for $d \in \mathbb{N}$ is $\mathbb{R}[x]_{\leq d}$. The set of vector-valued polynomials is $(\mathbb{R}[x])^n$ and the set of matrix-valued polynomials is $(\mathbb{R}[x])^{m \times n}$. The subset of $n \times n$ symmetric-matrix-valued polynomials is $\mathbb{S}^n[x]$, and its subcone of PSD (PD) polynomial matrices is $\mathbb{S}_+^n[x]$ ($\mathbb{S}_{++}^n[x]$). The set of SOS polynomials is $\Sigma[x]$, and the set of SOS matrices of size $n \times n$ is $\Sigma^n[x] \subset \mathbb{S}^n[x]$. The set of Weighted Sum of Squares (WSOS) polynomials over a BSA set \mathbb{K} is $\Sigma[\mathbb{K}]$, with WSOS matrices over the same set denoted as $\Sigma^n[\mathbb{K}]$.

The projection operator $\pi^x : (x, y) \mapsto x$ applied to a set $X \times Y$ is $\pi^x(X \times Y) = \{x \mid (x, y) \in X \times Y\}$.

2.3 Stability of Discrete-Time Linear Systems

Let $x_{t+1} = Ax_t + Bu_t$ be a discrete-time linear system. A system $A^{cl} = A + BK$ under the control law $u_t = Kx_t$ is *Schur* (stable) if all eigenvalues of $A + BK$ have absolute values less than 1. This subsection will provide certificates for the more conservative but computationally tractable notions of superstability and quadratic stability.

2.3.1 Superstability

The closed-loop system $A^{cl} = A + BK$ is superstable [11, 20] if,

$$\|A + BK\|_\infty < 1 \quad (L_\infty \text{ Operator Norm}). \quad (3)$$

Consequences of superstability are that $\|x\|_\infty$ is a polyhedral Control Lyapunov Function (CLF), and that each pole $a + bj$ of A^{cl} satisfies $|a| + |b| < 1$. Letting $\gamma = \|A + BK\|_\infty < 1$, a superstable system will satisfy $\|x_t\|_\infty \leq \gamma^{t/n} \|x_0\|_\infty$ for any initial condition x_0 to the closed loop system $A + BK$ [20]. Superstability in (3) may be realized by imposing an exponential $n(2^n)$ number of strict linear inequality constraints,

$$\sum_{j=1}^n (-1)_j^s A_{ij}^{cl} < 1 \quad \forall i = 1..n, s \in \{0, 1\}^n. \quad (4)$$

A more efficient method of imposing superstability is through applying convex lifts from [21]. A matrix $M \in \mathbb{R}^{n \times n}$ can be introduced satisfying,

$$\sum_{j=1}^n M_{ij} < 1 \quad \forall i = 1..n \quad (5a)$$

$$-M_{ij} \leq A_{ij}^{cl} \leq M_{ij} \quad \forall i, j = 1..n. \quad (5b)$$

There are $2n^2$ linear inequality and n strict linear inequality constraints in (5). If the closed loop matrix A^{cl} is superstable, such an M can be chosen as,

$$M_{ij} = |A_{ij} + \sum_{\ell=1}^m B_{i\ell} K_{\ell j}| \quad \forall i, j = 1..n \quad (6)$$

Superstability may be lost under a change-of-basis transformation of A^{cl} .

2.3.2 Quadratic Stabilizability

Superstability in the sense of [20] may be an overly restrictive criterion in finding a controller K . Quadratic stabilizability is another notion of stability that could be used when synthesizing control laws. A discrete-time system of the form in (1a) is quadratically stabilizable if there exists a symmetric positive definite matrix $Y \in \mathbb{S}_{++}^n$ and a matrix $S \in \mathbb{R}^{m \times n}$ such that [22],

$$P(A, B) = \begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \in \mathbb{S}_{++}^{2n}. \quad (7)$$

The controller K may be recovered as $K = SY^{-1}$. The matrix $P(A, B)$ is an affine function of (A, B) for fixed parameters (Y, S) , and Equation (7) is therefore an Linear Matrix Inequality (LMI). The function $x^T Y^{-1} x$ is a CLF.

Remark 1. *There is not necessarily a containment between the set of plants rendered superstable nor quadratically stable by a single state feedback control K .*

2.4 Semialgebraic Geometry and Sum of Squares

A Basic Semialgebraic (BSA) set is the locus of a finite number of bounded-degree polynomial inequality and equality constraints [1]. A representation exists for every BSA set \mathbb{K} can be represented as,

$$\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}, \quad (8)$$

for appropriate describing polynomials $\{g_i(x)\}_{i=1}^{N_g}$ and $\{h_j(x)\}_{j=1}^{N_h}$. The concatenation of describing polynomials implements the intersection of BSA sets. The projection $\pi^x : (x, y) \mapsto x$ as applied to a BSA set $\bar{\mathbb{G}}(x, y)$ is,

$$\mathbb{G}(x) = \pi^x \bar{\mathbb{G}}(x, y) = \{x \mid \exists y : (x, y) \in \bar{\mathbb{G}}\}. \quad (9)$$

The projection of a BSA set is not generally BSA, and is instead termed as a semialgebraic set (closure of BSA under projections, finite unions, and complements).

SOS methods yield certificates that a polynomial $p(x)$ is nonnegative over a semialgebraic set \mathbb{K} . A polynomial $p(x)$ is SOS ($p(x) \in \Sigma[x]$) if there exists a set of polynomials $\{q_i(x)\}_{i=1}^{N_q}$ such that $p(x) = \sum_{i=1}^{N_q} q_i(x)^2$. To each SOS certificate $\{q_i(x)\}_{i=1}^{N_q}$ there exists a nonunique choice of a size $s \in \mathbb{N}$, a Gram matrix $Q \in \mathbb{S}_+^s$, and a polynomial vector $v(x) \in \mathbb{R}[x]^s$ such that $p(x) = v(x)^T Q v(x)$. Given a matrix decomposition $Q = R^T R$, a valid certificate $\{q_i(x)\}_{i=1}^{N_q}$ may be recovered by $q(x) = Rv(x)$. In practice, $v(x)$ is often chosen to be the degree- d monomial map with $s = \binom{n+d}{d}$.

The set of SOS polynomials with degree at most $2d$ is $\Sigma[x]_{\leq 2d} \subset \mathbb{R}[x]_{\leq 2d}$. Optimization over the cone of nonnegative polynomials is generically NP-hard, but tightening to the SOS cone in fixed (n, d) yields more tractable problems that can be solved using SDPs [23]. By a theorem of Hilbert, the set of nonnegative polynomials in n variables will coincide with SOS polynomials with degree at most $2d$ only when $d = 1$ (quadratic), $n = 1$ (univariate), or $(n = 2, d = 2)$ (bivariate quartic) [24]. As the number of variables increases for fixed degree $d > 1$, the SOS cone is less and less able to describe the set of nonnegative polynomials (in the sense of volume ratios) [25].

SOS programming can be extended constrained optimization problems. The Putinar Positivstellensatz (Psatz) yields a certificate that a polynomial $p(x)$ is positive over a BSA \mathbb{K} [26],

$$p(x) = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x) + \sum_j \phi_j(x)h_j(x) \quad (10a)$$

$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma(x) \in (\Sigma[x])^{N_g}, \quad \phi \in (\mathbb{R}[x])^{N_h}. \quad (10b)$$

The cone of polynomials that admit a Putinar certificate of positivity over \mathbb{K} in the sense of (10) (where \mathbb{K} is described by g_i, h_j) will be defined as the WSOS cone $\Sigma[\mathbb{K}]$. The degree- $2d$ WSOS cone $\Sigma[\mathbb{K}]_{\leq 2d}$ is the set of polynomials where multipliers in (10b) are chosen such that each of the following products $(\sigma_0(x), \{\sigma_i(x)g_i(x)\}_{i=1}^{N_g}, \{\phi_j(x)h_j(x)\}_{j=1}^{N_h})$ has degree at most $2d$.

The BSA set $\mathbb{K} \subset \mathbb{R}^n$ is *compact* if there exists an $R \in (0, \infty)$ such that $\mathbb{K} \subseteq \{x \mid R - \|x\|_2^2 \geq 0\}$ (Heine-Borel). It is additionally *Archimedean* if an $R \in (0, \infty)$ exists such that $R - \|x\|_2^2 \in \Sigma[\mathbb{K}]$. Archimedeaness implies compactness, but there exists compact but non-Archimedean BSA sets [27]. Given knowledge of an R that proves compactness of \mathbb{K} , the compact set \mathbb{K} can be rendered Archimedean by adding the redundant constraint $R - \|x\|_2^2 \geq 0$ to \mathbb{K} 's description. The Putinar Psatz (10) is necessary and sufficient to prove that a polynomial $p(x)$ is positive when \mathbb{K} is Archimedean (Theorem 1.3 of [26]). Increasing the degree d of the WSOS cones $\Sigma[\mathbb{K}]_{\leq 2d}$ until a valid positivity certificate is found is known as the moment-SOS hierarchy [1]. The work in [28] analyzes the convergence rate of the moment-SOS hierarchy as d increases for POPs.

Each iteration of an Interior Point Methods solving an SDP with a matrix variable $X \in \mathbb{S}_+^N$ and M affine constraint requires $O(N^3 M + M^2 N^2)$ operations [29]. Certifying SOS-based nonnegativity of $p(x)$ over \mathbb{R}^n through the degree- d monomial map $v(x)$ requires a Gram Q matrix of size $N = \binom{n+d}{n}$ and $M = \binom{n+2d}{2d}$ coefficient matching constraints. The per-iteration Interior Point computational complexity for fixed n is therefore $O(d^{4n})$ and for fixed d is $O(n^{6d})$.

2.5 Polynomial Matrix Inequalities

Letting $P(x) \in \mathbb{S}^s[x]$ be a symmetric-matrix-valued polynomial, unconstrained and constrained Polynomial Matrix Inequalities (PMIs) posed over $P(x)$ are,

$$P(x) \succeq 0 \quad \forall x \in \mathbb{R}^n, \quad (11)$$

$$P(x) \succeq 0 \quad \forall x \in \mathbb{K}. \quad (12)$$

This subsection (and paper) will assume that the BSA set \mathbb{K} is described by a set of scalar inequality and equality constraints, and that the only matrix constraints involved are $P(x) \succeq 0$. Standard Putinar SOS methods may be used to certify the above PMIs through the use of scalarization. Defining $y \in \mathbb{R}^s$ as a new variable, constraint (11) is equivalent to imposing that the polynomial $y^T P(x) y$ is nonnegative for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^s$. Similarly, the constrained problem (12) can restrict $y^T P(x) y$ is nonnegative for all (x, y) in the compact set $\mathbb{K} \times \{y \mid \|y\|_2^2 = 1\}$. Applying scalarization increases the size of the Gram matrix used in the Putinar Psatz (10) to $\binom{n+s+d}{d}$.

The work in [30] introduced a Psatz for PMIs with a typically reduced Gram matrix size of $s \binom{n+d}{d}$. A matrix $P(x) \in \mathbb{S}[x]^{s \times s}$ is an SOS-matrix $P(x) \in \Sigma[x]^{s \times s}$ if there exists a size $s \in \mathbb{N}$, a polynomial vector

$v(x) \in \mathbb{R}[x]^q$, and a Gram matrix $Q \in \mathbb{S}_+^{q_s}$ such that (Lemma 1 of [30]),

$$P(x) = (v(x) \otimes I_q)^T Q (v(x) \otimes I_q). \quad (13)$$

SOS matrices and polynomials satisfy the relation $\Sigma[x] = \Sigma^1[x]$. The Scherer Psatz proving that a matrix $P(x)$ is PD ($P(x) \succ 0$) over $x \in \mathbb{K}$ (for some $\varepsilon > 0$) is (Theorem 2 of [30]),

$$P(x) = \sigma_0(x) + \sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x) + \varepsilon I_q \quad (14a)$$

$$\exists \sigma_0(x) \in \Sigma^s[x], \sigma_i(x) \in \Sigma^s[x], \phi_j \in \mathbb{S}^s[x]. \quad (14b)$$

The Scherer representation (14) is necessary and sufficient to certify $P(x) \succ 0$ over \mathbb{K} if \mathbb{K} is Archimedean (Remark 2 and Equation 9 of [30]). Note that the multipliers σ, μ from the Scherer Psatz (14b) are symmetric-matrix-valued polynomials of size s . The matrix-WSOS cone $\Sigma^s[\mathbb{K}]$ and its degree- $2d$ truncation $\Sigma^s[\mathbb{K}]_{\leq 2d}$ will denote the cone of matrices in $\mathbb{S}^n[\mathbb{K}]$ that admit Scherer positivity certificates in (14), just like $\Sigma[\mathbb{K}]$ and $\Sigma[\mathbb{K}]_{\leq 2d}$ in the scalar Putinar case. The SOS-matrix $\sigma_0(x)$ at the degree- $2d$ representation of (14) has a Gram matrix of size $s \binom{n+d}{d}$, and the matrix $\sigma_0(x)$ may be described by $s(s+1)/2 \binom{n+2d}{2d}$ independent coefficients. Given degrees $\forall i : d_i = \lceil \deg g_i(x) \rceil$, $d \geq d_i$, the multipliers $\sigma_i(x)$ are polynomial matrices with Gram matrices of size $s \binom{n+d-d_i}{d-d_i}$.

3 Full Program

This section will present SOS approaches towards recovering stabilizing controllers K (according to the criteria laid out in Section 2.3) applicable for all plants consistent with data in \mathcal{D} . In this section we set $\Delta u = 0$, $w = 0$ to simplify explanation and notation while still preserving the $A\Delta x$ bilinearity.

3.1 Consistency Sets

The BSA consistency set $\bar{\mathcal{P}}(A, B, \Delta x)$ of plants and noise values $(A, B, \Delta x) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times T}$ that are consistent with data \mathcal{D} under a noise bound of ϵ is described by (for ease of illustration, we assume that $\Delta u, w = 0$, a detailed discussion is provided in Section 7),

$$\bar{\mathcal{P}} : \left\{ \begin{array}{l} 0 = -\Delta x_{t+1} + A\Delta x_t + h_t^0 \quad \forall t = 1..T-1 \\ \|\Delta x_t\|_\infty \leq \epsilon \quad \forall t = 1..T \end{array} \right\}, \quad (15)$$

with an intermediate definition of the affine weights h^0 as,

$$h_t^0 = \hat{x}_{t+1} - A\hat{x}_t - Bu_t \quad \forall t = 1..T-1. \quad (16)$$

Remark 2. Multiple observations $\{\mathcal{D}_k\}_{k=1}^{N_d}$ of the same system may be combined together by BSA intersections to form $\bar{\mathcal{P}} = \bigcap_{k=1}^{N_d} \bar{\mathcal{P}}(\mathcal{D}_k)$.

The semialgebraic set of plants $\mathcal{P}(A, B)$ consistent with \mathcal{D} is the projection,

$$\mathcal{P}(A, B) = \pi^{A,B} \bar{\mathcal{P}}(A, B, \Delta x). \quad (17)$$

Remark 3. The consistency sets $\bar{\mathcal{P}}$ and \mathcal{P} may be nonconvex and could even be disconnected.

Remark 4. The describing constraints of $\bar{\mathcal{P}}$ are bilinear in terms of the groups (A, B) and (Δx) . Checking membership for fixed plant $(A_0, B_0) \in \mathcal{P}$ may be accomplished by solving a feasibility Linear Program (LP) in terms of Δx .

Problem 1. The data driven stabilization problem is to find K such that $A^{cl} = A + BK$ is [Superstable, Quadratically Stabilizable] for all $(A, B) \in \mathcal{P}$.

3.2 Function Programs

This section will pose problem 1 as a set of polynomial optimization programs, one for each class of stability. All programs will require the following assumption (for later convergence) The following assumption is required for finite convergence of Problem 2,

Assumption 1. *The sets $\bar{\mathcal{P}}$ (and therefore \mathcal{P}) are compact (Archimedean).*

Assumption 1 may be satisfied if sufficient data is collected.

3.2.1 Superstability

Superstabilization by a fixed $K \in \mathbb{R}^{m \times n}$ will be verified through equation (5) for all plants in \mathcal{P} . The M matrix in (5) will be a matrix-valued function $M(A, B) : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times n}$. The matrix function $M(A, B)$ will be Δx -independent given that the matrix $A + BK$ is also Δx -independent.

Problem 2. *Superstabilizing Problem 1 for a small margin $\delta > 0$ may be solved by,*

$$\underset{K \in \mathbb{R}^{m \times n}}{\text{find}} \quad \forall(A, B, \Delta x) \in \bar{\mathcal{P}}(A, B, \Delta x) : \quad (18a)$$

$$\forall i = 1..n : \quad (18b)$$

$$1 - \delta - \sum_{j=1}^n M_{ij}(A, B) \geq 0$$

$$\forall i = 1..n, j = 1..n : \quad (18c)$$

$$M_{ij}(A, B) - (A_{ij} + \sum_{\ell=1}^m B_{i\ell} K_{\ell j}) \geq 0$$

$$M_{ij}(A, B) + (A_{ij} + \sum_{\ell=1}^m B_{i\ell} K_{\ell j}) \geq 0$$

Lemma 3.1. *There exists a continuous selection for $M(A, B)$ given K under Assumption 1.*

Proof. Define $\mathcal{M} : \mathcal{P} \rightrightarrows \mathbb{R}_+^{n \times n}$ as the set-valued map (solution region to (18b)-(18c)),

$$\mathcal{M}(A, B) = \{M \in \mathbb{R}_+^{n \times n} \mid \forall i : \sum_j M_{ij} \leq 1 - \delta, \forall(i, j) : -M_{ij} \leq \pm(A_{ij} + \sum_{\ell=1}^m B_{i\ell} K_{\ell j})\}. \quad (19)$$

The right-hand sides of the constraints in (19) are each continuous (linear) functions of (A, B) . This ensures that \mathcal{M} is lower semi-continuous (Definition 1.4.2 in [31]) under the affine (continuous) changes in (A, B) in the compact domain \mathcal{P} by Theorem 2.4 in [32] (perturbations of right-hand-sides of linear-inequality-defined regions). Michael's theorem (Proposition 9.3.2 in [31]) suffices to show that a continuous selection of $M \in \mathcal{M}(A, B)$ exists, given that \mathcal{M} takes on closed convex values in the Banach space $\mathbb{R}^{n \times n}$, has a compact domain, and is lower-semicontinuous. One such continuous selection is the Minimal Map $M(A, B) = \operatorname{argmin}_{\tilde{M} \in \mathcal{M}(A, B)} \|\tilde{M}\|_F^2$. \square

Lemma 3.2. *There exists a $\delta' > \delta$ such that a polynomial $M^p(A, B)$ may be chosen for $M(A, B)$ given K .*

Proof. Let $\epsilon > 0$ be a tolerance such that $\forall(i, j) : \sup_{(A, B) \in \mathcal{P}} |(M_{ij}(A, B) + \epsilon) - M_{ij}^p(A, B)| \leq \epsilon$ by the Stone-Weierstrass theorem in the compact set \mathcal{P} [33]. This implies that $M^p(A, B) \geq \tilde{M}(A, B)$ everywhere in \mathcal{P} , because the residual $r_{ij}(A, B) = M_{ij}^p(A, B) - M_{ij}(A, B)$ takes on values between $[0, 2\epsilon]$. Now consider (18b) with M^p ,

$$1 - \delta - \sum_{j=1}^n M_{ij}^p = 1 - \delta - \sum_{j=1}^n (M_{ij} + r_{ij}). \quad (20a)$$

$$\geq 1 - \delta - 2\epsilon n - \sum_{j=1}^n (M_{ij}). \quad (20b)$$

For each row i , define Z_i^* as,

$$Z_i^* = \sup_{(A, B) \in \mathcal{P}} \sum_{j=1}^n (M_{ij}) \leq 1 - \delta, \quad (21)$$

From which it holds via (18b) that,

$$\forall i = 1..n : \quad 1 - \delta - Z_i^* \geq 0 \quad \implies \quad 1 - \delta/2 - Z_i^* > 0. \quad (22)$$

Substituting (22) into (20b) under the condition that (20b) must be nonnegative yields,

$$(1 - \delta/2 - Z_i^*) - 2\epsilon n \geq \delta/2 > 0 \quad (23)$$

Choosing $\delta' = \delta/2 + 2n\epsilon$ with $\epsilon < (1 - \delta/2)/(2n)$ (to ensure that $\delta' < 1$) will certify that M^p satisfies all inequality constraints w.r.t. δ' . □

3.2.2 Quadratic Stabilizability

Quadratic stabilizability of every plant in \mathcal{P} according to (7) will be enforced (if possible) by a common $K = MY^{-1}$,

Problem 3. A program for Quadratic Stabilization of Problem 1 is,

$$\begin{aligned} \underset{Y, M}{\text{find}} \begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \succ 0, & \quad \forall (A, B, \Delta x) \in \bar{\mathcal{P}} \\ Y \in \mathbb{S}_{++}^n, S \in \mathbb{R}^{m \times n} & \end{aligned} \quad (24)$$

The function $x^T Y^{-1} x$ is a common CLF for all plants in \mathcal{P} .

3.3 SOS Program and Numerical Considerations

Problems 2 and 3 may each be approximated by WSOS polynomials as discussed in Section 2.4 and 2.5.

3.3.1 Superstability SOS

SOS methods may be used to approximate the superstability Program of (18) by requiring that $M(A, B, \Delta x) \in (\mathbb{R}[A, B, \Delta x])^{n \times n}$ is a polynomial matrix of degree $2d$.

Define $q_i^{\text{row}}(A, B, \Delta x; K)$ as the left hand side of (18b), and let $q_{ij}^{\pm}(A, B, \Delta x; K)$ be the left hand side of each constraint in (18c). An example constraint from (18c) at (i, j) may be written as,

$$q_{ij}^+(A, B, \Delta x; K) = M_{ij}(A, B, \Delta x) - (A_{ij} + \sum_{\ell=1}^m B_{i\ell} K_{\ell j}).$$

Algorithm 1 expresses the degree- d WSOS tightening of Problem 2

Algorithm 1 Full Superstability Program

procedure SS FULL($d, \delta, \mathcal{D}, \epsilon$)

Solve (or find infeasibility certificate):

$$\underset{K, M}{\text{find}} K \in \mathbb{R}^{n \times m} \quad (25a)$$

$$M \in \mathbb{R}[A, B]_{\leq 2d} \quad (25b)$$

$$q_i^{\text{row}} \in \Sigma[\bar{\mathcal{P}}]_{\leq 2d} \quad \forall i \in 1..n \quad (25c)$$

$$q_{ij}^{\pm} \in \Sigma[\bar{\mathcal{P}}]_{\leq 2d} \quad \forall i, j \in 1..n \quad (25d)$$

return K

end procedure

There are $2n^2 + n$ nonnegativity constraints in Algorithm 1, each requiring a degree- $2d$ WSOS Psatz of (10). Each Psatz involves $n(n + m + T)$ variables $(A, B, \Delta x)$, which induces a Gram matrices of maximal size $\binom{n(n+m+T)+d}{d}$ at degree d .

Theorem 3.3. When all sets are Archimedian (assumption 1), Algorithm 1 will recover a superstabilizing K (if possible) solving as $d \rightarrow \infty$.

Proof. This theorem follows from results on convergence of POPs. Applying the moment-SOS hierarchy to a POP $p^* = \min_{x \in \mathbb{K}} p(x)$ for \mathbb{K} Archimedean will result in a convergent sequence of lower bounds $p_d^* \leq p_{d+1}^* \dots$ with $\lim_{d \rightarrow \infty} p_d^* = p^*$. The POP realization of nonnegativity program (18) is a feasibility problem with $p(x) = p^* = 0$. The lower bounds of the moment-SOS hierarchy will therefore take a value of $p_d^* = 0$ (feasible, superstabilizing K found) or $p_d^* = -\infty$ (dual infeasible, superstabilizing K not found). By the limit property of $\lim_{d \rightarrow \infty} p_d^* = p^*$ under the Archimedean assumption 1, a superstabilizing K will be found if possible as the degree d increases. \square

3.4 Quadratic SOS

The quadratic stabilizability SOS program is described in Algorithm 2.

Algorithm 2 Full Quadratic Stabilizability Program

procedure QUAD FULL($d, \delta, \mathcal{D}, \epsilon$)
Solve (or find infeasibility certificate):

$$\underset{Y, S}{\text{find}} \begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \in \Sigma^{2n}[\bar{\mathcal{P}}]_{\leq 2d}$$

$$Y \in \mathbb{S}_{++}^n, S \in \mathbb{R}^{m \times n} \quad (26)$$

return $K = SY^{-1}$
end procedure

Problem (26) involves a PMI constraint of for a matrix of size $2n$ and a PD constraint of size n . The PMI contains $n(n + m + T)$ variables $(A, B, \Delta x)$. The maximal Gram matrix size as induced by the the degree- d Scherer Psatz (14b) is $2n \binom{n+m+T+d}{d}$.

Theorem 3.4. *Algorithm 2 will recover a Quadratically stabilizing K (if one exists) as $d \rightarrow \infty$ under the Archimedean assumption 1.*

Proof. This follows from the proof of Theorem 3.3 as modified for the Scherer Psatz in [30]. \square

4 Alternatives Program

This section will formulate and use a Matrix Theorem of Alternatives in order to reduce the computational expense of running Algorithms 1 and 2. The cost savings are derived from elimination of the affine-entering noise variables Δx .

4.1 Theorem of Alternatives

Let $q : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \rightarrow \mathbb{S}^s$ be a symmetric-matrix valued function satisfying the constraint,

$$q(A, B) \in \mathbb{S}_{++}^s \quad \forall (A, B) \in \mathcal{P}. \quad (27)$$

If constraint (27) is satisfied (feasible), then the following problem is infeasible:

$$\underset{(A, B) \in \mathcal{P}}{\text{find}} -\lambda_{\min}(q(A, B)) \geq 0 \quad (28)$$

Lemma 4.1. *Constraints (27) and (28) are strong alternatives (either one or the other is feasible).*

Proof. If (27) is feasible, then $\lambda_{\min} q(A, B)$ is positive for all $(A, B) \in \mathcal{P}$. Therefore there does not exist an (A, B) such that $\lambda_{\min}(A, B) \leq 0$, which is the statement of (28). On the opposite side, feasibility of (28) with an $(A', B') : \lambda_{\min}(q(A', B')) \leq 0$ implies that $q(A', B') \notin \mathbb{S}_{++}^s$ and therefore (27) is infeasible. Additionally, there is no case where (27) and (28) are both infeasible: either all $q(A, B)$ are PD (27) or there exists a non-PD counterexample (28). \square

A set of dual variable functions $(\mu(A, B), \zeta(A, B))$ may be defined based on the constraint description from (15),

$$\mu_{ti} : \mathcal{P} \rightarrow \mathbb{S}^s \quad \forall i = 1..n, t = 1..T-1 \quad (29a)$$

$$\zeta_{ti}^{\pm} : \mathcal{P} \rightarrow \mathbb{S}_+^s \quad \forall i = 1..n, t = 1..T. \quad (29b)$$

The multipliers from (29), the Scherer Psatz (14), and the Robust Counterpart method of [34] can be used to form the weighted sum $\Phi(A, B; \zeta^{\pm}, \mu) : \mathcal{P} \rightarrow \mathbb{S}^s$,

$$\begin{aligned} \Phi &= -q(A, B) + \sum_{t=1, i=1}^{T, n} (\zeta_{ti}^+(\epsilon - \Delta x_t) + \zeta_{ti}^-(\epsilon + \Delta x_{t,i})) \\ &\quad + \sum_{t=1, i=1}^{T-1, n} \mu_{ti}(-\Delta x_{t+1, i} + A_i \Delta x_{ti} + h_{ti}^0) \end{aligned} \quad (30)$$

The dual multipliers will always be treated as (possibly nonunique and discontinuous) functions $\mu(A, B)$ or $\zeta(A, B)$, but their (A, B) dependence may be omitted to condense notation.

Theorem 4.2. *A sufficient condition for infeasibility of program (28) is if there exists multipliers ζ^{\pm}, μ according to (29) such that,*

$$\forall (A, B) \in \mathcal{P} : \sup_{\Delta x \in \mathbb{R}^{n \times T}} \lambda_{\max}(\Phi(A, B; \zeta^{\pm}, \mu)) < 0. \quad (31)$$

Proof. This theorem holds by arguments from [34, 35] with modifications for the matrix case.

Any point $(A, B, \Delta x) \in \bar{\mathcal{P}}$ must satisfy $\|\Delta x_t\|_{\infty} \leq \epsilon$ for all times $t = 1..T$ (15). This implies that $\epsilon \pm \Delta x_t$ are nonnegative vectors for each time t for $\Delta x \in \pi^{\Delta x} \bar{\mathcal{P}}$, and therefore $\zeta_{ti}^{\pm}(\epsilon \pm \Delta x_{ti})$ are each PSD matrices given that ζ_{ti}^{\pm} are PSD. Additionally, the data consistency constraints in $\Delta x_{t+1} + A \Delta x_t + h_t^0$ from (15) evaluate to 0 for $(A, B, \Delta x) \in \bar{\mathcal{P}}$, so μ times these zero quantities result in a zero matrix. The addition of these PSD and Zero multiplier terms to $-q$ ensures that $\lambda_{\max}(\Phi) \geq \lambda_{\max}(-q)$. Finding multipliers (ζ^{\pm}, μ) such that (31) holds therefore implies that $-q$ is Negative Definite (q is PD) over the space \mathcal{P} . This definiteness statement certifies infeasibility of (28), because there cannot exist a negative eigenvalue of q as (A, B) ranges over \mathcal{P} . □

Theorem 4.3. *Theorem 4.2 is additionally necessary for infeasibility of (28) in the case where $s = 1$ ($q(A, B)$ is scalar).*

Proof. The term $\lambda_{\min}(q)$ is replaced by q in the scalar case of (28). The constraints in (15) are affine in Δx , which means they are both convex and concave in the variable Δx . The term $q(A, B)$ is also independent of Δx . Concavity of the constraints in Δx is enough to establish necessity and strong alternatives by convex duality (Section 5.8 of [35]). □

Remark 5. *Refer to [36] for an example where the Alternatives procedure (31) with $s > 1$ is sufficient but not necessary (robust SDPs over interval matrices: “computationally tractable conservative approximation”). The work in [37] formulates robust SDPs involving polytopic uncertainty as a generally intractable two-stage optimization program.*

Equation (33) can be simplified and transformed into a feasibility program by explicitly defining and constraining its Δx -supremal value. The sum Φ is affine in Δx , and the constant terms (in Δx) of Φ are $Q(A, B; \zeta^{\pm}, \mu)$,

$$Q = -q(A, B) + \sum_{t=1, i=1}^{T, n} \epsilon (\zeta_{ti}^+ + \zeta_{ti}^-) + \sum_{t=1, i=1}^{T-1, n} \mu_{ti} h_{ti}^0 \quad (32)$$

The sum Φ may therefore be expressed as,

$$\Phi = Q + \sum_{t=1, i=1}^{T-1, n} \mu_{ti} A_i \Delta x_{ti} - \sum_{t=2, i=1}^{T, n} \mu_{t-1, i} \Delta x_{ti} + \sum_{t=1, i=1}^{T, n} (\zeta_{ti}^- - \zeta_{ti}^+) \Delta x_{ti} \quad (33)$$

The supremal value of Φ in (31) given $(A, B; \zeta^\pm, \mu)$ is,

$$\sup_{\Delta x} \lambda_{max}(\Phi) = \begin{cases} \lambda_{max}(Q) & \zeta_{1i}^+ - \zeta_{1i}^- = \sum_{j=1}^n A_{ji} \mu_{1j} \\ & \zeta_{ti}^+ - \zeta_{ti}^- = \sum_{j=1}^n A_{ji} \mu_{tj} - \mu_{t-1,i} \\ & \zeta_{Ti}^+ - \zeta_{Ti}^- = -\mu_{T-1,i} \\ \infty & \text{else} \end{cases} \quad (34)$$

A feasibility program that ensures $\sup_{\Delta x} \lambda_{max}(\Phi(A, B)) < 0$, $\forall (A, B) \in \mathcal{P}$ (ranging over $i = 1..n$) is,

$$\underset{\zeta, \mu}{\text{find}} \quad Q(A, B; \zeta^\pm, \mu) < 0 \quad \forall (A, B) \in \mathcal{P} \quad (35a)$$

$$\zeta_{1i}^+ - \zeta_{1i}^- = \sum_{j=1}^n A_{ji} \mu_{1j} \quad (35b)$$

$$\zeta_{ti}^+ - \zeta_{ti}^- = \sum_{j=1}^n A_{ji} \mu_{tj} - \mu_{t-1,i} \quad \forall t \in 2..T-1 \quad (35c)$$

$$\zeta_{Ti}^+ - \zeta_{Ti}^- = -\mu_{T-1,i} \quad (35d)$$

$$\zeta_{ti}^\pm(A, B) \in \mathbb{S}_+^s \quad \forall t = 1..T, \quad (35e)$$

$$\mu_{ti}(A, B) \in \mathbb{S}^s \quad \forall t = 1..T-1 \quad (35f)$$

Theorem 4.4. *When $s = 1$, program (35) is equivalent to the statement in (27), and is a strong alternative to (28). When $s > 1$, program (35) is a weak alternative to (28).*

Proof. Lemma 4.1 proves that (27) and (28) are strong alternatives.

In the case where $s = 1$, Theorem 4.3 proves that (28) and (31) are strong alternatives. Given that (35) is an explicit condition for validity of (31), it holds that (35) and (28) are strong alternatives and therefore (27) and (35) are equivalent.

In the more general case where $s > 1$, Theorem 4.2 proves that (28) and (31) are weak alternatives. Successfully finding a certificate (ζ^\pm, μ) from (35) validates that (28) is infeasible. It is not possible for both (35) and (28) to hold simultaneously. However, there may still exist cases where (27) holds but (35) is infeasible. \square

Theorem 4.5. *The dual multipliers $\zeta_{ti}^\pm(A, B)$ and $\mu_{ti}(A, B)$ which certify that $q(A, B) \succ 0$ over \mathcal{P} via (35) may be chosen to be continuous.*

Proof. See Appendix A. \square

Theorem 4.6. *Whenever (35) is feasible, $\zeta_{ti}^\pm(A, B)$ and $\mu_{ti}(A, B)$ may additionally be required to be polynomial matrices.*

Proof. See Appendix B. \square

4.2 Alternatives SOS

The degree- d WSOS truncation of program (35) to certify positive definiteness in (27) is contained in Algorithm 3. The Alternatives Psatz in Algorithm 3 requires the following assumption to ensure convergence as $d \rightarrow \infty$,

Assumption 2. *An Archimedean set $\Pi(A, B) \supseteq \mathcal{P}$ is previously known.*

Remark 6. *A set Π may arise from prior knowledge about plant behavior and its reasonable limits (e.g. (A, B) are contained in a known box).*

Remark 7. *All of the equality constraints in (36d)-(36f) are linear constraints in the coefficients of ζ^\pm, μ .*

Remark 8. *Choosing ζ of degree $2d$ and μ of degree $2d - 1$ ensures that the expression of Φ remains degree $2d$ when $\deg q \leq 2d$.*

Algorithm 4 is the Alternatives WSOS formulation for the Full Superstabilizing Algorithm 1. Similarly, Algorithm 5 is the Alternatives WSOS formulation for the Full Quadratically Stabilizing Algorithm 2.

Algorithm 3 Alternatives Psatz ($\Sigma_{\leq 2d}^{s, \text{alt}}[\mathcal{P}]$)

procedure ALTERN PSATZ($d, q(A, B), \Pi, \mathcal{D}, \epsilon, s$)
Solve (or find infeasibility certificate):

$$\zeta_{ti}^{\pm}(A, B) \in \Sigma^s[\Pi]_{\leq 2d} \quad (36a)$$

$$\mu_{ti}(A, B) \in \mathbb{S}^s[A, B]_{\leq 2d-1} \quad (36b)$$

$$-Q(A, B; \zeta^{\pm}, \mu) \in \Sigma[\Pi]_{\leq 2d} \text{ (from (32))} \quad (36c)$$

$$\zeta_{1i}^+ - \zeta_{1i}^- = \sum_{j=1}^n A_{ji} \mu_{1j} \quad (36d)$$

$$\zeta_{ti}^+ - \zeta_{ti}^- = \sum_{j=1}^n A_{ji} \mu_{tj} - \mu_{t-1, i} \quad \forall t \in 2..T-1 \quad (36e)$$

$$\zeta_{Ti}^+ - \zeta_{Ti}^- = -\mu_{T-1, i}. \quad (36f)$$

return ζ, μ (or Infeasibility)
end procedure

Algorithm 4 Alternatives Superstability Program

procedure SS ALTERN($d, \delta, \mathcal{D}, \epsilon, \Pi$) Solve (or find infeasibility certificate):

$$K \in \mathbb{R}^{n \times m} \quad (37a)$$

$$M_{ij} \in \mathbb{R}[A, B]_{\leq 2d} \quad \forall i, j \in 1..n \quad (37b)$$

$$q_i^{\text{row}} \in \Sigma_{\leq 2d}^{1, \text{alt}}[\Pi] \quad \forall i \in 1..n \quad (37c)$$

$$q_{ij}^{\pm} \in \Sigma_{\leq 2d}^{1, \text{alt}}[\Pi] \quad \forall i, j \in 1..n \quad (37d)$$

return K (or Infeasibility)
end procedure

5 Worst-Case-H2-Optimal Control

The previously presented Full and Alternative methods perform superstabilization or quadratic stabilization by attempting to find a single feasible controller K . This section generalizes existing LMIs for discrete-time worst-case H_2 -optimal control over all possible linear systems in the consistency set \mathcal{P} . The discrete-time linear system model used in this section has a state $x \in \mathbb{R}^n$, a control input $u \in \mathbb{R}^m$, an exogenous input $w \in \mathbb{R}^e$ and matrix $E \in \mathbb{R}^{n \times e}$, and a regulated output $z \in \mathbb{R}^r$ defined by matrices $C \in \mathbb{R}^{r \times n}$, $D \in \mathbb{R}^{r \times m}$,

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + Ew_t \\ z_t &= Cx_t + Du_t. \end{aligned} \quad (39)$$

The matrices (C, D, E) will be a-priori given, while a consistency set description $(A, B) \in \mathcal{P}$ is available through the input-state collected data \mathcal{D} . The original LMIs referenced in this section are compiled in [38] and utilize a convexification method from [39].

The following LMI to minimize the H_2 -norm between $w \rightarrow z$ is expressed in Section 4.2.2 of [38].

Problem 4. *There exists a static output feedback $u = Kx$ such that the H_2 -norm of the system (39) is*

Algorithm 5 Alternatives Quadratic Stability Program

procedure QS ALTERN($d, \delta, \mathcal{D}, \epsilon, \Pi$) Solve (or find infeasibility certificate):

$$\begin{aligned} \text{find}_{Y,M} \begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \in \Sigma^{2n, \text{alt}}[\Pi]_{\leq 2d} \\ Y \in \mathbb{S}_{++}^n, S \in \mathbb{R}^{m \times n} \end{aligned} \quad (38)$$

return $K = SY^{-1}$ (or Infeasibility)
end procedure

upper bounded by a value $\gamma \in \mathbb{R}^+$ if and only if the following LMIs are feasible,

$$\text{find}_{Y,Z,S} \begin{bmatrix} Y - EE^T & AY + BS \\ * & Y \end{bmatrix} \in \mathbb{S}_{++}^{2n} \quad (40a)$$

$$\begin{bmatrix} Z & CY + DS \\ * & Y \end{bmatrix} \in \mathbb{S}_{++}^{n+r} \quad (40b)$$

$$\text{Tr}(Z) \leq \gamma^2 \quad (40c)$$

$$Y \in \mathbb{S}_{++}^n, Z \in \mathbb{S}_+^r, S \in \mathbb{R}^{m \times n}. \quad (40d)$$

The corresponding feedback gain is $K = SY^{-1}$.

Problem (40) may be posed as a worst-case H_2 problem by minimizing γ^2 and ensuring that the LMIs hold over all possible plants in the consistency set \mathcal{P} .

Problem 5. *There exists a static output feedback $u = Kx$ such that the worst-case optimal H_2 -norm of the system (39) over a set $(A, B) \in \mathcal{P}$ may be upper bounded by,*

$$\begin{aligned} \inf_{\gamma \in \mathbb{R}^+, Y, Z, S} \quad & \gamma^2 \quad (41) \\ & \begin{bmatrix} Y - EE^T & AY + BS \\ * & Y \end{bmatrix} \in \mathbb{S}_{++}^{2n} \quad \forall (A, B) \in \mathcal{P} \\ & \begin{bmatrix} Z & CY + DS \\ * & Y \end{bmatrix} \in \mathbb{S}_+^{n+r} \\ & \gamma^2 - \text{Tr}(Z) \geq 0 \\ & Y \in \mathbb{S}_{++}^n, Z \in \mathbb{S}_+^r, S \in \mathbb{R}^{m \times n} \end{aligned}$$

The feedback gain $K = SY^{-1}$ is constant in (A, B) .

A degree- $2d$ Alternatives problem may be posed for worst-case H_2 synthesis by restricting the top LMI in (40) (holding over $\forall (A, B) \in \mathcal{P}$) to the cone $\Sigma^{2n, \text{alt}}[\Pi]_{\leq 2d}$. All SOS problems will be presented for the Alternatives case due to the Gram large matrix sizes involved in the Full algorithms. The Full problem would restrict the same matrix to the cone $\Sigma^{2n}[\bar{\mathcal{P}}]_{\leq 2d}$.

6 Computational Complexity

This section tabulates the computational complexity involved in executing superstabilization or quadratically stabilization algorithms 1, 2, 4, or 5 under measurement noise. As a reminder, the Full methods involve $p_F = n(n + m + T)$ variables $(A, B, \Delta x)$ while the Alternatives methods employ $p_A = n(n + m)$ variables (A, B) after eliminating Δx .

The following tables will use the notation $\mathbb{R}(\cdot)$ for the size (cone) of a vector $\mathbb{R}^{(\cdot) \times 1}$ (free variable) and $\mathbb{S}_+(\cdot)$ for the size of a matrix $\mathbb{S}_+^{(\cdot)}$ (semidefinite variable).

Algorithm 6 Worst-Case H_2 Program (Alternative)

procedure H_2 ALTERN($d, \mathcal{D}, \epsilon, C, D$)
 Solve (or find infeasibility certificate):

$$\begin{aligned}
 & \min_{\gamma \in \mathbb{R}^+, Y, Z, S} \quad \gamma^2 & (42) \\
 & \begin{bmatrix} Y - EE^T & AY + BS \\ * & Y \end{bmatrix} \in \Sigma^{2n, \text{alt}}[\Pi]_{\leq 2d} \\
 & \begin{bmatrix} Z & CY + DS \\ * & Y \end{bmatrix} \in \mathbb{S}_+^{n+r} \\
 & \gamma^2 - \text{Tr}(Z) \geq 0 \\
 & Y \in \mathbb{S}_{++}^n, Z \in \mathbb{S}_+^r, S \in \mathbb{R}^{m \times n}
 \end{aligned}$$

return $H_2 = \gamma, K = SY^{-1}$ (or Infeasibility)
end procedure

Table 1: Size of Superstabilizing Method (Psatz)

	q	σ_0	σ_i	μ_j
Number of Polynomials	1	1	$2nT$	$2n(T-1)$
Full	$\mathbb{R} \binom{p_F+2d}{2d}$	$\mathbb{S}_+ \binom{p_F+d}{d}$	$\mathbb{S}_+ \binom{p_F+d-1}{d-1}$	$\mathbb{R} \binom{p_F+2d-2}{2d-2}$
Alternatives	$\mathbb{R} \binom{p_A+2d}{2d}$	$\mathbb{S}_+ \binom{p_A+d}{d}$	$\mathbb{S}_+ \binom{p_A+d}{d}$	$\mathbb{R} \binom{p_A+2d-1}{2d-1}$

The superstabilizing algorithms 1 and 4 involve $2n^2+n$ polynomials $q(A, B, \Delta x)$ that must be nonnegative. Table 1 compares the variable sizes in the Putinar Psatz (10) and the Alternatives Psatz (36).

The quadratically stabilizing programs in algorithm 2 and 5 involve a single PMI of size $2n$, along with a positive definite matrix $Y \in \mathbb{S}_{++}^n$. Let $\nu = n(n+1)/2$ be the number of free variables in a symmetric matrix in \mathbb{S}^n . Table 2 lists the size of the matrices involved in the quadratically stabilizing Psatz expressions (14) and (36).

Table 2: Size of Quadratically Stabilizing Method (Psatz)

	q	σ_0	σ_i	μ_j
Number of Polynomials	1	1	$2nT$	$2n(T-1)$
Full	$\mathbb{R} \nu \binom{p_F+2d}{2d}$	$\mathbb{S}_+ 2n \binom{p_F+d}{d}$	$\mathbb{S}_+ 2n \binom{p_F+d-1}{d-1}$	$\mathbb{R} \nu \binom{p_F+2d-2}{2d-2}$
Alternatives	$\mathbb{R} \nu \binom{p_A+2d}{2d}$	$\mathbb{S}_+ 2n \binom{p_A+d}{d}$	$\mathbb{S}_+ 2n \binom{p_A+d}{d}$	$\mathbb{R} \nu \binom{p_A+2d-1}{2d-1}$

Remark 9. *The dual variables μ have a degree of $2d-2$ in the Full formulation ($A\Delta x$) and a degree of $2d-1$ in the alternatives formulation.*

The main reductions in computational complexity are,

1. The Alternatives Gram matrix sizes are independent with respect T .
2. Experimental results demonstrate that the Full method often only works with $d \geq 2$ while the Alternatives methods can permit convergence with $d = 1$.

Table 3 and 4 displays the size (but not multiplicities) of variables involved in the Superstabilizing Algorithms 1, 4 and quadratically stabilizing Algorithms 2, 5 for parameters of $n = 2, m = 1, d_{full} = 2, d_{altern} = 1$ and increasing T .

Table 3: Size of Variables for Superstability

	q	σ_0	σ_i	μ_j
Alternatives	28	7	7	7
Full($T = 4$)	3060	120	15	120
Full($T = 6$)	7315	190	19	190
Full($T = 8$)	14950	276	23	276

Table 4: Size of Variables for Quadratic Stability

	q	σ_0	σ_i	μ_j
Alternatives	280	28	28	70
Full($T = 4$)	30600	480	60	1200
Full($T = 6$)	73150	760	76	1900
Full($T = 8$)	149500	1104	92	2760

Remark 10. *Quadratic stability is much more costly to enforce compared with superstability. The full method quickly becomes intractable as the number of samples increases.*

The H_2 optimal control program in Algorithm 6 has the same PSD matrix sizes as in Table 4 for the Scherer constraint. It is additionally required that $Y \in \mathbb{S}_{++}^n$ and $Z \in \mathbb{S}_+^r$, but these sizes of n and r are small in comparison to the sizes in Table 4.

7 All Noise Sources

This section reinserts the process noise w and input noise Δu from the model (1) into the description of plant consistency sets.

7.1 Set Description and Full

Let $(\epsilon_x, \epsilon_u, \epsilon_w) \geq 0$ be L_∞ bounds for the measurement, input, and process noise respectively. The BSA consistency set associated with the data $\mathcal{D} = (\tilde{x}_t, \tilde{u}_t)$ is,

$$\bar{\mathcal{P}}^{\text{all}} : \left\{ \begin{array}{l} \|\Delta x_t\|_\infty \leq \epsilon_x, \quad \forall t = 1..T \\ \|\Delta u_t\|_\infty \leq \epsilon_u, \quad \forall t = 1..T - 1 \\ \|w_t\|_\infty \leq \epsilon_w, \quad \forall t = 1..T - 1 \end{array} \right\}, \quad (43)$$

with the defined quantities for all $t = 1..T - 1$,

$$0 = -\Delta x_{t+1} + A\Delta x_t + B\Delta u_t + w_t + h_t^{\text{all}} \quad (44a)$$

$$h_t^{\text{all}} = \tilde{x}_{t+1} - A\tilde{x}_t - B\tilde{u}_t. \quad (44b)$$

The set $\bar{\mathcal{P}}^{\text{all}}$ in (43) is described by $2((T - 1)(2n + m) + n)$ polynomial inequality constraints and $n(T - 1)$ linear equality constraints in terms of the $p_F^{\text{all}} = n(n + m) + (T - 1)(2n + m) + n$ variables $(A, B, \Delta x, \Delta u, w)$. Just as in (15) with \mathcal{P} and $\bar{\mathcal{P}}$, the semialgebraic set of consistent plants $\mathcal{P}^{\text{all}}(A, B)$ may be formed by the projection,

$$\mathcal{P}^{\text{all}}(A, B) = \pi^{A,B} \bar{\mathcal{P}}^{\text{all}}(A, B, \Delta x, \Delta u, w). \quad (45)$$

The combination of measurement, input, and process noise may be incorporated into the Full algorithms 1 (superstability) and 2 (quadratic stability) by imposing Putinar (10) positivity constraints over the set $\bar{\mathcal{P}}^{\text{all}}$ in (43).

Remark 11. We note that structured process noise $x_{k+1} = Ax_k + Bu_k + Ew_k$ for $E \in \mathbb{R}^{n \times e}$ may be incorporated into the All-noise framework. Additionally, the process noise variables w may be eliminated when $e \leq n$. Define a left inverse E^\dagger with $E^\dagger E = I_e$ and a matrix \mathcal{N} containing a basis for the nullspace of E^T in its columns. The following equations may be imposed to represent the process noise constraints,

$$\begin{aligned} \epsilon_w &\geq \|E^\dagger ((\tilde{x}_{t+1} - \Delta x_{t+1}) - A(\tilde{x}_t - \Delta x_t) - B\tilde{u}_t)\|_\infty & \forall t = 1..T-1 \\ 0 &= \mathcal{N}((\tilde{x}_{t+1} - \Delta x_{t+1}) - A(\tilde{x}_t - \Delta x_t) - B\tilde{u}_t) & \forall t = 1..T-1. \end{aligned}$$

7.2 Alternatives

The BSA set (43) is described by $n(n+m) + T(2n+m) - n$ variables $(A, B, \Delta x, \Delta u, w)$. The variables $(\Delta x, \Delta u, w)$ may be eliminated by following the Theorem of Alternatives laid out in Section 4. The process noise variables w_t under the constraint $\forall t = 1..T-1$, $\|w_t\|_\infty \leq \epsilon_w$ will be eliminated by rearranging terms in (2),

$$\begin{aligned} w_t &= A(\hat{x}_t - \Delta x_t) + B(u_t - \Delta u_t) - (\hat{x}_{t+1} - \Delta x_{t+1}) \\ &= A\Delta x_t + B\Delta u_t + h_t^{\text{all}} - \Delta x_{t+1}. \end{aligned} \quad (46)$$

A certificate of PSD-ness for the following matrix function $q(A, B)$ will be derived (from (27)),

$$q(A, B) \in \mathbb{S}_{++}^s \quad \forall (A, B) \in \mathcal{P}^{\text{all}}. \quad (47)$$

Dual variables $\mu^\pm \in (\mathbb{S}_+^s[A, B])^{n \times (T-1)}$ for ϵ_w , $\psi^\pm \in (\mathbb{S}_+^s[A, B])^{n \times T}$ for ϵ_u and $\zeta^\pm \in (\mathbb{S}_+^s[A, B])^{n \times T}$ for ϵ_x may be initialized according to the Putinar multipliers (10b) to form the weighted sum Φ^{all} ,

$$\begin{aligned} \Phi^{\text{all}} &= -q(A, B) + \sum_{t=1, i=1}^{T, n} (\zeta_{ti}^+(\epsilon_x - \Delta x_t) + \zeta_{ti}^-(\epsilon_x + \Delta x_t)) \\ &+ \sum_{t, i}^{T, m} (\psi_{ti}^+(\epsilon_u - \Delta u_t) + \psi_{ti}^-(\epsilon_u + \Delta u_t)) \\ &+ \sum_{t, i}^{T-1, n} \mu_{ti}^-(\epsilon_w - (A_i \Delta x_{ti} + B_i \Delta u_{ti} + h_{ti}^{\text{all}} - \Delta x_{t+1, i})) \\ &+ \sum_{t, i}^{T-1, n} \mu_{ti}^+(\epsilon_w + (A_i \Delta x_{ti} + B_i \Delta u_{ti} + h_{ti}^{\text{all}} - \Delta x_{t+1, i})). \end{aligned} \quad (48)$$

The $(\Delta x, \Delta u)$ -constant terms of Φ^{all} is,

$$\begin{aligned} Q^{\text{all}} &= -q(A, B) + \sum_{t=1, i=1}^{T-1, n} \epsilon_w (\mu^- + \mu^+) \\ &+ \sum_{t=1, i=1}^{T-1, n} h_{ti}^{\text{all}} (\mu^+ - \mu^-) \\ &+ \sum_{t=1, i=1}^{T, m} \epsilon_u (\psi_{ti}^+ + \psi_{ti}^-) + \sum_{t=1, i=1}^{T, n} \epsilon_x (\zeta_{ti}^+ + \zeta_{ti}^-). \end{aligned} \quad (49)$$

Define the following symbol $\Delta\zeta = \zeta^+ - \zeta^-$, with a similar structure holding for $\Delta\psi$ and $\Delta\mu$. Following the supremum ≤ 0 procedure of Section 4 leads to an alternatives-based nonnegativity certificate of (47),

$$\underset{\zeta, \mu, \psi}{\text{find}} \quad -Q^{\text{all}}(A, B; \zeta^\pm, \mu^\pm, \psi^\pm) \in \mathbb{S}_+^s \quad \forall (A, B) \quad (50a)$$

$$\Delta\zeta_{1i} = \sum_{j=1}^n A_{ji} (\mu_{1j}^+ - \mu_{1j}^-) \quad (50b)$$

$$\Delta\psi_{ti} = B\Delta\mu_{ti} \quad \forall t = 1..T \quad (50c)$$

$$\Delta\zeta_{ti} = \sum_{j=1}^n A_{ji} \Delta\mu_{tj} - \Delta\mu_{t-1, i} \quad \forall t \in 2..T-1 \quad (50d)$$

$$\Delta\zeta_{Ti} = -\Delta\mu_{T-1, i} \quad (50e)$$

$$\psi_{ti}^\pm \in \mathbb{S}_+^s[A, B] \quad \forall i = 1..n, t \in 1..T \quad (50f)$$

$$\zeta_{ti}^\pm \in \mathbb{S}_+^s[A, B] \quad \forall i = 1..n, t \in 1..T \quad (50g)$$

$$\mu_{ti}^\pm \in \mathbb{S}_+^s[A, B] \quad \forall i = 1..n, t \in 1..T-1. \quad (50h)$$

The certificate (50) involves only the $n(n+m)$ variables (A, B) at the cost of requiring $2T(2n+m) - 2n + 1$ Scherer Psatz constraints in (A, B) ((50a) and (50f)-(50h)). The cone of matrix-valued polynomials that

admit certificates in (50) may be expressed as $\Sigma^s[A, B]^{\text{alt,all}}$. The cone $\Sigma^n[A, B]_{\leq d}^{\text{alt,all}}$ may be substituted in for $\Sigma^s[A, B]_{\leq d}^{\text{alt}}$ in the Alternatives Algorithms 4 (superstability) and 5 (quadratic stability). An Archimedean set $\Pi^{\text{all}} \supseteq \mathcal{P}^{\text{all}}$ must be known to ensure convergence of the Alternatives certificate as the degree $d \rightarrow \infty$ (from Assumption 1). All Scherer constraints would then take place under $-Q, \mu_{ti}^{\pm}, \zeta_{ti}^{\pm}, \psi_{ti}^{\pm} \in \Sigma^s[\Pi^{\text{all}}]_{\leq 2d}$ at degree- d .

8 Numerical Examples

MATLAB (2021a) code to generate the below examples is publicly available¹. Dependencies include Mosek [40] and YALMIP [41].

In this section, we first discuss the performance of the quadratic stabilization algorithm as a function of noise level and number of samples with the Monte Carlo experiments. One example is presented where superstabilization algorithm fails while quadratic stabilization algorithm works. Next, we show the worst-case optimal control for both H_2 and H_∞ performance. A followup experiment shows the result where all type of noises are considered. Finally, we illustrate that the partial information helps identify a controller.

8.1 Monte Carlo Simulations for Stabilization

To test the reliability of the proposed method, we collected 50 trajectories with different level of noise and u, x_0 uniformly distributed in $[-1, 1]$. Then we apply the Algorithm 4 and 5 on the following open-loop unstable system:

$$A = \begin{bmatrix} 0.6863 & 0.3968 \\ 0.3456 & 1.0388 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4170 & 0.0001 \\ 0.7203 & 0.3023 \end{bmatrix} \quad (51)$$

We first investigate the effect of noise by fixing $T = 8$. The result is reported in TABLE 5 showing the percentage of successful design for superstability (SS) and quadratic stability (QS).

Table 5: Success rate (%) as a function of ϵ with $T = 8$

ϵ	0.05	0.08	0.11	0.14
SS	100	84	66	34
QS	100	100	80	58

As expected, QS performs better than SS since it is a less restrictive stability condition. Increasing the noise level expands the consistency set, which in turn renders the problem of finding a single stabilizing controller more difficult. Collecting more sample data at the same noise bound $\epsilon = 0.14$ reduces the size of the consistency set, as illustrated in TABLE 6.

Table 6: Success rate as a function of T with $\epsilon = 0.14$

T	8	10	12	14
SS	34	54	70	86
QS	58	72	90	98

To see the advantage of QS over SS, we now consider a simple mass-spring-damper system shown below with $x_1 = x, x_2 = \dot{x}, u = F$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad (52)$$

This system is not superstabilizable since any state feedback $u = Kx$ cannot affect the first row of A , hence the infinity norm of $A + BK$ is always greater or equal to 1. However, one can easily apply the Algorithm 5 to find a quadratically stabilizing controller.

¹https://github.com/jarmill/error_in_variables

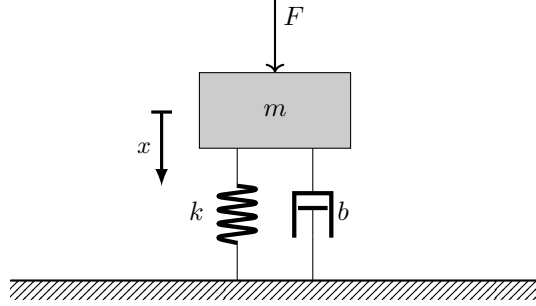


Figure 1: Illustration of a Spring-Mass-Damper System with parameters in (52)

8.2 Worst-Case-Optimal Control

To analyze the optimal performance of the proposed method, we first solve the standard H_2 problem (40) ($C = I_{r \times n}$, $D = [\mathbf{0}_{n,m}; I_m]$, $E = I_{n \times e}$), with known A, B defined by (51). We denote the benchmark as $\gamma_2 = 1.9084$. Now we apply Algorithm 6 to 50 trajectories with different level of noise. The effect of noise is shown in TABLE 7 with fixed $T = 8$.

For each of the 50 trajectories, Algorithm 6 returns a control policy K and a worst-case- H_2 upper bound $\gamma_{2,worst}$ (valid for all $(A, B) \in \mathcal{P}$). The quantity $\gamma_{2,clp}$ (closed-loop poles) is the H_2 norm found by applying K as an input to the ground-truth system and solving Problem 4. It therefore holds that $\gamma_{2,worst} \geq \gamma_{2,clp}$ for each trajectory. The quantities returned in Table 7 and all subsequent tables are the median values of $(\gamma_{2,clp}, \gamma_{2,worst})$ over the 50 trajectories in order to prevent outliers deviating results.

Table 7: H_2 performance as a function of ϵ with $T = 8$

ϵ	0.05	0.08	0.11	0.14
$\gamma_{2,clp}$	1.9684	2.0715	2.1773	2.1456
$\gamma_{2,worst}$	2.3004	2.7308	3.2279	4.3137

As we increase the noise, $\gamma_{2,worst}$ also increases since the consistency set is expanded. However, $\gamma_{2,clp}$ does not necessarily increase since we only optimize in the worst case. It is also worth noting that $\gamma_2 \leq \gamma_{2,clp} \leq \gamma_{2,worst}$ and the equality holds only if there is no noise. H_2 performance can be improved by collecting more samples as shown in TABLE 8.

Table 8: H_2 performance as a function of T with $\epsilon = 0.08$

T	8	10	12	14
$\gamma_{2,clp}$	2.0715	1.9637	1.9373	1.9321
$\gamma_{2,worst}$	2.7308	2.4160	2.2328	2.2014

8.3 All Noise

The proposed framework is easily extended to handle different type of noises which better captures the estimation error of the collected data. Consider the following set of noise bounds

$$\begin{aligned}
 \text{set1} : & \epsilon_x = 0.03, \epsilon_u = 0.00, \epsilon_w = 0.00 \\
 \text{set2} : & \epsilon_x = 0.00, \epsilon_u = 0.02, \epsilon_w = 0.00 \\
 \text{set3} : & \epsilon_x = 0.00, \epsilon_u = 0.00, \epsilon_w = 0.05 \\
 \text{set4} : & \epsilon_x = 0.03, \epsilon_u = 0.02, \epsilon_w = 0.00 \\
 \text{set5} : & \epsilon_x = 0.00, \epsilon_u = 0.02, \epsilon_w = 0.05 \\
 \text{set6} : & \epsilon_x = 0.03, \epsilon_u = 0.02, \epsilon_w = 0.05
 \end{aligned} \tag{53}$$

For system (51), we collected a single data trajectory of length $T = 8$ starting from an initial state of $x_1 = [1; 0]$ with u uniformly distributed in $[-1, 1]^2$.

Table 9: H_2 performance for different sets of noise

set	1	2	3	4	5	6
$\gamma_{2,clp}$	1.9340	1.9131	1.9750	1.9615	2.1249	2.1659
$\gamma_{2,worst}$	2.0681	1.9628	2.1294	2.1554	2.5029	2.5973

The H_2 norm for the nominal plant is $\gamma_2 = 1.9084$. It is clear that adding more type of noise expands the consistency set hence leads to a larger worst-case H_2 norm.

8.4 Partial Information

It is easy to incorporate the partial information in the proposed framework. Instead of treating all entries of A, B as unknown variables, we can suppose q entries of (A, B) are known. There are now $n(n+m) - q$ free variables defining the consistency set, producing a smaller Gram matrix of $\binom{n(n+m)-q+d}{d}$ as compared to $\binom{n(n+m)+d}{d}$ and ensuring that it is easier to find a quadratically stabilizing K with better performance both theoretically and computationally. For instance, if we assume that the first row of A is known and apply Algorithm 6 with $T = 8$, $\epsilon = 0.08$, we get $\gamma_{2,clp} = 1.9568$, $\gamma_{2,worst} = 2.2566$ as compared to $\gamma_{2,clp} = 2.0715$, $\gamma_{2,worst} = 2.7308$ in the first column of TABLE 8.

9 Extensions

This section outlines extensions to the proposed SOS-based stabilization framework under measurement noise.

9.1 Time-Varying Noise Sets

The measurement noise Δx in $\bar{\mathcal{P}}$ satisfies an L_∞ constraint of $\|\Delta x_t\| \leq \epsilon$. In principle, the Full method will converge to a superstabilizing or quadratically stabilizing controller when each Δx_t is restricted to be a member of an Archimedean BSA set. Such sets may vary in time or even in the plant parameters (A, B) so long as $\bar{\mathcal{P}}$ remains Archimedean. The Alternatives framework may be employed to simplify computational complexity when the sets \mathcal{F}_t with $\Delta x_t \in \mathcal{F}_t$, $\forall t = 1..T$ are restricted to be polytopes.

9.2 Non-uniform Sampling

The polynomial-optimization-based framework for EIV control may be performed in cases with missing state data. This extension will pose consistency sets with $(w, \Delta u) = 0$ for ease of notation and derivation. Consider a trajectory with samples $(\hat{x}_t, \hat{x}_{t+2})$ and inputs (u_t, u_{t+1}) in which the state observation \hat{x}_{t+1} is missing. Treating x_{t+1} as an unknown variable would add a new source of uncertainty to the description of $\bar{\mathcal{P}}$ and increase the complexity of posing Psatz constraints. The unknown state x_{t+1} can be eliminated using the relation $x_{t+1} = Ax_t + Bu_t$:

$$x_{t+2} = Ax_{t+1} + Bu_{t+1} \quad (54a)$$

$$x_{t+2} = A(Ax_t + Bu_t) + Bu_{t+1} \quad (54b)$$

$$x_{t+2} = A^2x_t + ABu_t + Bu_{t+1} \quad (54c)$$

$$\hat{x}_{t+2} - \Delta x_{t+2} = A^2(\hat{x}_t - \Delta x_t) + ABu_t + Bu_{t+1}. \quad (54d)$$

Given the state observations $(\hat{x}_t, \hat{x}_{t+r})$ for $r \in \mathbb{N}, r > 0$ and the input history $u_{k=k=t}^{t+r}$, the Δx -affine expression for the missing data case is,

$$-\Delta x_{t+r} + A^r \Delta x_t + (\hat{x}_{t+r} - \sum_{k=0}^r A^{r-k} Bu_{t+k}) = 0. \quad (55)$$

Assume that state measurements are taken at N_s times $\{t_s\}_{s=1}^{N_s} \subseteq (1..T)$. The missing-data records \mathcal{D}^{md} is comprised of the inputs $\{u_t\}_{t=1}^T$ and states $\{\hat{x}_{t_s}\}_{s=1}^{N_s}$. The missing-data consistency set is,

$$\bar{\mathcal{P}}^{md} : \left\{ \begin{array}{l} \text{Eq (55) with } t = t_s, r = t_{s+1} - t_s \quad \forall k = 1..N_s - 1 \\ \|\Delta x_{t_k}\|_\infty \leq \epsilon \quad \forall k = 1..N_s \end{array} \right\}, \quad (56)$$

The Full Algorithms 1 and 2 may be used directly for missing-data by employing the consistency set $\bar{\mathcal{P}}^{md}$ in (56) rather than $\bar{\mathcal{P}}$.

An Alternatives Psatz from 3 of a function $q(A, B) \in \mathbb{S}_{++} \forall (A, B) \in \bar{\mathcal{P}}^{md}$ may also be derived. The Superstability Alternatives program for missing-data involves two multipliers ζ_s^\pm for each sample $s \in 1, \dots, N_s$. Each sample s from $1..N_s - 1$ inspires a multiplier μ_s and an $h_k^0 = \hat{x}_{t_{s+1}} - \sum_{k=0}^r A^{r-k} B u_{t_s+k}$ from (55). The Superstability Psatz from (35) with non-uniform sampling is,

$$\exists \zeta_s^\pm \geq 0, \mu_s \quad (57a)$$

$$q(A, B) \geq \sum_{s,i} \epsilon (\zeta_{s,i}^+ + \zeta_{s,i}^-) + \sum_{s=1}^{N_s-1} \mu_s^{N_s} h_s^0 \quad \forall (A, B) \quad (57b)$$

$$\zeta_1^+ - \zeta_1^- = (A^{t_2-t_1})^T \mu_1 \quad (57c)$$

$$\zeta_s^+ - \zeta_s^- = (A^{t_{s+1}-t_s})^T \mu_s - \mu_{s-1} \quad \forall s \in 1..N_s - 1 \quad (57d)$$

$$\zeta_{N_s}^+ - \zeta_{N_s}^- = -\mu_{N_s-1} \quad (57e)$$

Algorithm 4 may be used to perform superstabilizing control with the Psatz of (57). A similar derivation can take place for Alternatives-based quadratic stabilization with missing-data.

9.3 Switched Systems Stabilization

The final extension focuses on stabilization programs of switched systems. Let there be a collection of N_s discrete-time linear subsystems,

$$x_{t+1} = A_s x_t + B_s u_t \quad \forall s = 1..N_s. \quad (58)$$

The switching sequence of the system (58) is a function $S : 1..T \rightarrow 1..N_s$ denoting the resident subsystem at time t . An execution of the switched system (58) is a pair $\mathcal{D}_S = (\{(x_t, u_t)\}_{t=1}^T, \{S_t\}_{t=1}^{T-1})$ such that,

$$x_{t+1} = A_{S_t} x_t + B_{S_t} u_t \quad \forall t = 1..T - 1. \quad (59)$$

The execution \mathcal{D}_S is *labeled* if the switching sequence S_t is known. A labeled execution \mathcal{D}_S corrupted by ϵ -bounded measurement noise Δx inspires the following consistency set in terms of the unknown subsystem plants $\{A_s \in \mathbb{R}^{n \times n}, B_s \in \mathbb{R}^{n \times m}\}_{s=1}^T$ and measurement noise $\Delta x \in \mathbb{R}^{n \times T}$,

$$\bar{\mathcal{P}}_S : \left\{ \begin{array}{l} \forall t = 1..T - 1 : \\ \hat{x}_{t+1} - \Delta x_{t+1} = A_{S_t} (\hat{x}_t - \Delta x_t) + B_{S_t} u_t \\ \forall t = 1..T : \quad \|\Delta x_t\|_\infty \leq \epsilon \end{array} \right\}. \quad (60)$$

A control K that superstabilizes each subsystem (A_s, B_s) may be found by solving Algorithm 1 in which all constraints are posed over the set (60), yielding a maximum PSD constraint of size $\binom{N_s n(n+m) + nT + d}{d}$ per Psatz. Correlatively sparse cliques $(\Delta x_t, \Delta x_{t+1}, A_s, B_s)$ may be developed for each $t = 1..T - 1$ and $s = 1..N_s$ [42], with $N_s(T - 1)$ instances of PSD constraints of maximal size $\binom{n(n+m+2)+d}{d}$ per Psatz. The Theorem of Alternatives may also be applied in order to eliminate the noise terms Δx , but the resultant polynomials (ζ^\pm, μ) would have arguments consisting of all plant parameters $\{(A_s, B_s)\}_{s=1}^{N_s}$ with $2nT + 1$ PSD constraints of maximal size $\binom{N_s n(n+m)+d}{d}$ per Psatz.

10 Conclusion

This paper formulated a state-feedback stabilization problem for systems corrupted by L_∞ -bounded measurement, process, and input noise. WSOS programs for superstabilizability and quadratic stabilizability (Full) will converge to their respective controllers (if such a controller exists) as the degree tends towards infinity. Such programs are computationally expensive with regards to the size of the PSD matrices required in SDPs. A theorem of alternatives was deployed to create equivalent (superstability) and conservative (quadratic stabilizability) programs at a reduced computational cost by eliminating the noise variables. This framework was extended to the robust control of plants in the consistency set, as laid out through H_2 methods.

Future work involves developing static-output-feedback and dynamic-output-feedback in the case of combined input noise and measurement noise. Other work involves analyzing conditions for which the sets $\bar{\mathcal{P}}$ and \mathcal{P} are compact in terms of the collected data \mathcal{D} (e.g. sampling complexity and defining a concept of persistency of excitation), and quantifying the conservativeness involved with utilizing the Theorem of Alternatives in the $s > 1$ case. Tractability of this method would improve with further development and reductions in complexity of solving SDPs.

A Continuity of Multipliers

This appendix proves that the multiplier functions $(\zeta^\pm(A, B), \mu(A, B))$ from program (35) (when feasible) may be chosen to be continuous for any PD function $q(A, B)$ over \mathcal{P} (Theorem 4.5). This proof will establish that a set-valued map based on a variant of the feasible set of (35) is lower semicontinuous, and will then apply Michael's theorem to certify continuous selections.

Let $q(A, B) : \mathcal{P} \rightarrow \mathbb{S}_{++}^s$ be a function that is PD over the consistency set of plants \mathcal{P} with a certificate of nonnegativity $(\zeta^\pm(A, B), \mu(A, B))$ by program (35). We note that \mathcal{P} is compact by Assumption 1 and that the mapping from \mathcal{P} to the constraints in (35) are affine (Lipschitz) in (A, B) over the compact \mathcal{P} .

Let $Z = (\mathbb{S}^s)^{n \times T} \times (\mathbb{S}^s)^{n \times T} \times (\mathbb{S}^s)^{n \times (T-1)}$ be the residing space (possible range) of the multipliers $(\zeta^+(A, B), \zeta^-(A, B), \mu(A, B))$. In this appendix, the notation $(\zeta^\pm(A, B), \mu(A, B))$ will refer to functions from (35), and variables (ζ^\pm, μ) lacking arguments (A, B) will be values in Z .

A convex-set-valued map $S : \mathcal{P} \rightrightarrows Z$ may be defined as the feasible set of program (35) for each $(A, B) \in \mathcal{P}$. The domain of S is equal to \mathcal{P} , since the functions $(\zeta^\pm(A, B), \mu(A, B))$ have values in Z ($S(A, B) \neq \emptyset$) for all $(A, B) \in \mathcal{P}$. The range is nonclosed due to the PD constraint in (35a).

Define $\tau^* = \min_{(A, B) \in \mathcal{P}} \lambda_{\min}(q(A, B))$ as the minimal possible eigenvalue of the PD matrix $q(A, B)$. We note that the minimum is attained with $\tau^* > 0$ because \mathcal{P} is compact and $\lambda_{\min}(q(A, B))$ is a continuous function of (A, B) (given that the eigenvalues of a matrix are continuous in the matrix entries).

We define the closed-convex-valued map $S^\tau : \mathcal{P} \rightrightarrows Z$ for a value $0 < \tau < \tau^*$ as the set of solutions $(\zeta^\pm, \mu) \in Z$ for each $(A, B) \in \mathcal{P}$ to,

$$-Q(A, B; \zeta^\pm, \mu) - \tau I \in \mathbb{S}_+^s \quad (61a)$$

$$\zeta_{1i}^+ - \zeta_{1i}^- = \sum_{j=1}^n A_{ji} \mu_{1j} \quad (61b)$$

$$\zeta_{ti}^+ - \zeta_{ti}^- = \sum_{j=1}^n A_{ji} \mu_{tj} - \mu_{t-1, i} \quad \forall t \in 2..T-1 \quad (61c)$$

$$\zeta_{Ti}^+ - \zeta_{Ti}^- = -\mu_{T-1, i} \quad (61d)$$

$$\zeta_{ti}^\pm \in \mathbb{S}_+^s, \mu_{ti} \in \mathbb{S}^s \quad \forall t = 1..T, \quad (61e)$$

The set-valued mappings S and S^τ are related by $S^\tau \subset S$ for all admissible τ .

Any (possibly discontinuous) solution function $(\zeta^\pm(A, B), \mu(A, B))$ that certifies positive-definiteness of $q(A, B)$ over \mathcal{P} via (35) satisfies,

$$(\zeta^\pm(A, B), \mu(A, B)) \in S^\tau(A, B) \quad \forall (A, B) \in \mathcal{P}, \forall \tau \in (0, \tau^*). \quad (62)$$

A solution $(\zeta^\pm, \mu) \in S^\tau(A, B)$ is a *Slater point* if all matrices in (61a) and (61e) are PD and all equality constraints are fulfilled. A solution $(\zeta^\pm, \mu) \in S^{\tau'}(A, B)$ for some $\tau' > 0$ may be transformed into a new

solution $(\tilde{\zeta}^\pm, \mu) \in S^{\tau/4}(A, B)$ such that $(\tilde{\zeta}^\pm, \mu)$ is a Slater point. Specifically, we express $-Q - \tau I$ from (32) as,

$$-Q - \tau I = q(A, B) - \sum_{t=1, i=1}^{T, n} \epsilon (\zeta_{ti}^+ + \zeta_{ti}^-) - \sum_{t=1, i=1}^{T-1, n} \mu_{ti} h_{ti}^0 - \tau I \quad (63a)$$

$$= q(A, B) - \sum_{t=1, i=1}^{T, n} \epsilon (\zeta_{ti}^+ + \zeta_{ti}^- + \frac{\tau}{2Tn\epsilon} I) - \sum_{t=1, i=1}^{T-1, n} \mu_{ti} h_{ti}^0 - \frac{\tau}{2} I, \quad (63b)$$

The shifted multipliers $\tilde{\zeta}^\pm$ may be defined as,

$$\tilde{\zeta}_{ti}^\pm = \zeta_{ti}^\pm + \frac{\tau}{4Tn\epsilon} I, \quad (63c)$$

which yields,

$$-Q - \tau I = q(A, B) - \sum_{t=1, i=1}^{T, n} \epsilon (\tilde{\zeta}_{ti}^+ + \tilde{\zeta}_{ti}^-) - \sum_{t=1, i=1}^{T-1, n} \mu_{ti} h_{ti}^0 - \frac{\tau}{2} I = -\tilde{Q} - \frac{\tau}{2} I. \quad (63d)$$

Additionally, the differences in (61b)-(61d) cancel out with $\forall t, i$:

$$\tilde{\zeta}_{ti}^+ - \tilde{\zeta}_{ti}^- = \left(\zeta_{ti}^+ + \frac{\tau}{4Tn\epsilon} \right) - \left(\zeta_{ti}^- + \frac{\tau}{4Tn\epsilon} \right) = \zeta_{ti}^+ - \zeta_{ti}^- \quad (64)$$

Lemma A.1. *The solution $(\tilde{\zeta}^\pm(A, B), \mu(A, B))$ constructed from a certificate $(\zeta(A, B), \mu(A, B))$ from (35) by (63c) is a Slater point of $S^{\frac{\tau}{4}}(A, B)$ for each $(A, B) \in \mathcal{P}$.*

Proof. Given that $\zeta_{ti}^\pm(A, B)$ is PSD for each (t, i) (from (35e)), adding a PD matrix $\frac{\tau}{4Tn\epsilon} I$ to $\zeta_{ti}^\pm(A, B)$ will produce a PD $\tilde{\zeta}_{ti}^\pm(A, B)$ from (63c). The matrix $-\tilde{Q}$ from (63d) is also PD, since $-\tilde{Q} - \frac{\tau}{2} I \succeq 0 \implies -\tilde{Q} \succ \frac{\tau}{4} I$. All equality constraints remain feasible by the observation in (64), fulfilling the Slater point description. \square

Lemma A.2. *The set-valued mapping S^τ is lower-semicontinuous over \mathcal{P} .*

Proof. This follows from the (strong) Slater point characterization points in S^τ within $S^{\frac{\tau}{4}}$ from Lemma (A.1) by arguments from [43] extended to the Matrix case, given that S^τ has closed convex images and sends a compact set to a Banach space. \square

The condition for Michael's theorem (Thm. 9.1.2 of [31]) hold guaranteeing a continuous selection of S^τ is that \mathcal{P} is compact, Z is a Banach space, S^τ is lower-semicontinuous, and S^τ has closed, nonempty, convex images for each $(A, B) \in \mathcal{P}$. All of these conditions hold, so a continuous selection $(\zeta_s^\pm, \mu_s) : \mathcal{P} \rightarrow Z$ may be chosen with $(\zeta_s^\pm(A, B), \mu_s(A, B)) \in S^\tau(A, B) \subset S(A, B)$. These continuous functions (ζ_s^\pm, μ_s) may therefore be used to certify positive-definiteness of $q(A, B)$ over \mathcal{P} in (35).

B Polynomial Approximability of Multipliers

This appendix proves that a PD function $q(A, B)$ over \mathcal{P} may be certified using polynomial multipliers $(\tilde{\zeta}^\pm(A, B), \tilde{\mu}(A, B))$ whenever (35) is feasible (Theorem 4.6). The proof will proceed through the introduction of three positive approximation tolerances $(\eta_0, \eta_1, \eta_2) > 0$ for use in the Stone-Weierstrass theorem over the compact set \mathcal{P} . For a matrix $F \in \mathbb{R}^{n \times m}$, define the element-wise maximum-absolute-value operator as $\mathbf{Mabs}(F) = \max_{(i,j)} |F_{ij}|$.

Let $(\zeta^\pm(A, B), \mu(A, B))$ be a continuous multiplier certificate from (35), in which continuity was proven by Appendix A. A symmetric polynomial multiplier matrix $\tilde{\mu}$ can be created using the Stone-Weierstrass theorem as,

$$\forall (t, i) : \tilde{\mu}_{ti}(A, B) \in \mathbb{S}^s[A, B] \quad \sup_{(A, B) \in \mathcal{P}} \mathbf{Mabs}(\tilde{\mu}_{ti}(A, B) - \mu_{ti}(A, B)) < \eta_0. \quad (65)$$

B.1 Multiplier Bound

This subsection will approximate the ζ multipliers by polynomials.

Let $\gamma_{ti} \in \mathbb{S}[A, B]$ be the right-hand-sides of constraints (35b)-(35d) given $\tilde{\mu}$ with $(\forall i = 1..n)$,

$$\gamma_{1i} = \sum_{j=1}^n A_{ji} \tilde{\mu}_{1j} \quad (66a)$$

$$\gamma_{ti} = \sum_{j=1}^n A_{ji} \tilde{\mu}_{tj} - \tilde{\mu}_{t-1,i} \quad \forall t \in 2..T-1 \quad (66b)$$

$$\gamma_{Ti} = -\tilde{\mu}_{T-1,i}. \quad (66c)$$

The equations $\zeta_{ti}^+ - \zeta_{ti}^- = \gamma_{ti}$ from constraints (35b)-(35d) have solutions that can be parameterized by a set of continuous symmetric-matrix-valued functions $\phi_{ti}(A, B) : \mathcal{P} \rightarrow \mathbb{S}^s$,

$$\forall(t, i) : \quad \zeta_{ti}^+(A, B) = \phi_{ti}(A, B)/2 + \gamma_{ti}(A, B)/2 \quad (67a)$$

$$\zeta_{ti}^-(A, B) = \phi_{ti}(A, B)/2 - \gamma_{ti}(A, B)/2. \quad (67b)$$

The functions ϕ_{ti} may be η_1 -approximated by polynomials in the compact region \mathcal{P} :

$$\forall(t, i) : \quad \tilde{\phi}_{ti}(A, B) \in \mathbb{S}^s[A, B] \quad \sup_{(A, B) \in \mathcal{P}} \mathbf{Mabs}(\tilde{\phi}_{ti}(A, B) - \phi_{ti}(A, B)) < \eta_1. \quad (68)$$

A tolerance $\eta_2 > 0$ is introduced to define the polynomial approximators $\tilde{\zeta}$,

$$\forall(t, i) : \quad \tilde{\zeta}_{ti}^+(A, B) = \tilde{\phi}_{ti}(A, B)/2 + \gamma_{ti}(A, B)/2 + (\eta_2/2)I \quad (69a)$$

$$\tilde{\zeta}_{ti}^-(A, B) = \tilde{\phi}_{ti}(A, B)/2 - \gamma_{ti}(A, B)/2 + (\eta_2/2)I. \quad (69b)$$

The approximants $\tilde{\zeta}$ are related to the original multipliers ζ by,

$$\forall(t, i) : \quad \tilde{\zeta}_{ti}^\pm(A, B) = \zeta_{ti}^\pm(A, B) + (\tilde{\phi}_{ti}(A, B) - \phi_{ti}(A, B))/2 + (\eta_2/2)I. \quad (70)$$

The approximant $\tilde{\zeta}$ must take on PSD values in order to ensure that it is a valid multiplier for (35e). We utilize a result from the theory of interval matrices in order to choose η_2 .

Lemma B.1. *Let $M \in \mathbb{S}_{++}^s$ and $R \in \mathbb{S}^s$ be matrices with $\mathbf{Mabs}(R) \leq \eta$ for some $\eta > 0$. A sufficient condition for $M + R \in \mathbb{S}_{++}^s$ for all possible choices of R is that $M - \eta s I \in \mathbb{S}_{++}^s$.*

Proof. A symmetric interval matrix G^I may be described by a center $C \in \mathbb{S}^s$ and a radius $\Delta \in \mathbb{S}^s$, in which the elements $G \in G^I$ satisfy $G_{ij} \in [C_{ij} - \Delta_{ij}, C_{ij} + \Delta_{ij}]$ and $G_{ij} = G_{ji}$. Letting $\rho(\Delta)$ be the spectral radius of Δ (maximum absolute value of eigenvalues), Theorem 5 of [44] states that a sufficient condition for all symmetric interval matrices in the family G^I to be PD is that $\lambda_{\min}(C) > \rho(\Delta)$.

Let J_s be the all-ones matrix of size s . The interval matrix $(M + R)$ for $\mathbf{Mabs}(R) \leq \eta$ has a center of $C = M$ and a radius of $\Delta = \eta J_s$. The spectral radius of ηJ_s is $\rho(\eta J_s) = \eta s$. The sufficient condition for Interval PD-ness as provided by Theorem 5 of [44] is that $\lambda_{\min}(M) > \eta s$, or equivalently $M - \eta s I \in \mathbb{S}_{++}^s$. \square

The matrix $(\eta_2/2)I - (\tilde{\phi}_{ti}(A, B) - \phi_{ti}(A, B))/2$ from (70) may be treated as an interval matrix with center $C = (\eta_2/2)I$ and radius $\delta = (J_s \eta_1)/2$. Given that $\zeta_{ti}^\pm \in \mathbb{S}_+^s(A, B)$, a sufficient condition for $\tilde{\zeta}_{ti}^\pm \in \mathbb{S}_+^s(A, B)$ by Lemma B.1 is,

$$\eta_2 > \eta_1 s. \quad (71)$$

B.2 Certificate Bound

The Δx -constant term Q in (32) has a polynomial approximation (when substituting $\zeta^\pm \rightarrow \tilde{\zeta}^\pm$, $\mu \rightarrow \tilde{\mu}$),

$$\tilde{Q} = -q(A, B) + \sum_{t=1, i=1}^{T, n} \epsilon \left(\tilde{\zeta}_{ti}^+ + \tilde{\zeta}_{ti}^- \right) + \sum_{t=1, i=1}^{T-1, n} \tilde{\mu}_{ti} h_{ti}^0 \quad (72a)$$

$$= Q + \sum_{t=1, i=1}^{T, n} \epsilon (\tilde{\phi}_{ti} - \phi_{ti} + \eta_2 I) + \sum_{t=1, i=1}^{T-1, n} (\tilde{\mu}_{ti} - \mu_{ti}) h_{ti}^0 \quad (72b)$$

$$= Q + \epsilon \eta_2 T n I + \sum_{t=1, i=1}^{T, n} \epsilon (\tilde{\phi}_{ti} - \phi_{ti}) + \sum_{t=1, i=1}^{T-1, n} (\tilde{\mu}_{ti} - \mu_{ti}) h_{ti}^0 \quad (72c)$$

$$-\tilde{Q} = -Q - \epsilon \eta_2 T n I - \sum_{t=1, i=1}^{T, n} \epsilon (\tilde{\phi}_{ti} - \phi_{ti}) - \sum_{t=1, i=1}^{T-1, n} (\tilde{\mu}_{ti} - \mu_{ti}) h_{ti}^0. \quad (72d)$$

Interval matrices and Lemma B.1 will be used to derive a sufficient condition on (η_0, η_1, η_2) such that $-\tilde{Q}$ is PD for all $(A, B) \in \mathcal{P}$ (satisfying condition (35a)). Define the smallest eigenvalue of $-Q$ as,

$$\lambda^* = \min_{(A, B) \in \mathcal{P}} \lambda_{\min}(-Q(A, B)) > 0 \quad (73)$$

Because Q satisfies (35a), $-Q$ is PD over \mathcal{P} and therefore $\lambda^* > 0$. Further define \bar{h}^0 using (16) as,

$$\bar{h}_{ti}^0 = \max_{(A, B) \in \mathcal{P}} h_{ti}^0, \quad \forall (t, i). \quad (74)$$

The expression in (72d) therefore satisfies,

$$-\tilde{Q} \succeq (\lambda^* - \epsilon \eta_2 T n) I - \sum_{t=1, i=1}^{T, n} \epsilon (\tilde{\phi}_{ti} - \phi_{ti}) - \sum_{t=1, i=1}^{T-1, n} (\tilde{\mu}_{ti} - \mu_{ti}) \bar{h}_{ti}^0. \quad (75)$$

Each $(\tilde{\phi}_{ti} - \phi_{ti})$ term in (75) may be treated as an interval matrix with radius $\mathbf{0}$ and center $\eta_1 J_s$. Likewise, each $(\tilde{\mu}_{ti} - \mu_{ti})$ may be generalized to an interval matrix with radius $\mathbf{0}$ and center $\eta_0 J_s$. This interval matrix description of the right-hand-side of (75) leads to an interval matrix with center C_Q and radius Δ_Q ,

$$C_Q = (\lambda^* - \epsilon \eta_2 T n) I \quad (76a)$$

$$\Delta_Q = \sum_{t=1, i=1}^{T, n} \epsilon (\eta_1 J_s) + \sum_{t=1, i=1}^{T-1, n} (\eta_0 J_s) \bar{h}_{ti}^0 \quad (76b)$$

The intermediate definition of $\bar{H} = \sum_{t=1, i=1}^{T-1, n} \bar{h}_{ti}^0$ leads to,

$$\Delta_Q = (T n \epsilon J_s) \eta_1 + (\bar{H} J_s) \eta_0. \quad (76c)$$

The spectral radius of Δ_Q is,

$$\rho(\Delta_Q) = (T n \epsilon s) \eta_1 + (\bar{H} s) \eta_0. \quad (77)$$

By Lemma B.1, a sufficient condition for the interval family in (76) to be PD is if,

$$(\lambda^* - (T n \epsilon) \eta_2) I - [(T n \epsilon s) \eta_1 + (\bar{H} s) \eta_0] I \in \mathbb{S}_{++}^s, \quad (78)$$

which may be equivalently expressed as,

$$\lambda^* > (T n \epsilon) \eta_2 + (T n \epsilon s) \eta_1 + (\bar{H} s) \eta_0 > 0. \quad (79)$$

The combined conditions for admissible (η_0, η_1, η_2) are,

$$\eta_0, \eta_1, \eta_2 > 0 \quad (80a)$$

$$\eta_2 > s \eta_1 \quad (80b)$$

$$\lambda^* > (T n \epsilon) \eta_2 + (T n \epsilon s) \eta_1 + (\bar{H} s) \eta_0. \quad (80c)$$

One possible choice of $(\eta_0, \eta_1, \eta_2) > 0$ satisfying (80) is,

$$\eta_0 = \frac{\lambda^*}{4 \bar{H} s} \quad \eta_1 = \frac{\lambda^*}{2 T n \epsilon (2s + 1)} \quad \eta_2 = \frac{\lambda^* (s + 1)}{2 T n \epsilon (2s + 1)}. \quad (81)$$

A polynomial multiplier certificate (ζ^\pm, μ) will therefore always exist whenever (35) is satisfied.

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