Analysis and Control of Input-Affine Systems Using Parameterized Robust Counterparts

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Motivation

Problem Ingredients

Dynamics that are affine in input $w(\cdot)$

$$\dot{x}(t) = f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x) \tag{1}$$

Uncertainty/input is constrained in convex set (e.g. box):

$$w(t) \in W(t, x(t)) \tag{2}$$

Use input-affinity and convex-sets to simplify calculations

Problems to look at

Relevant dynamical systems problems:

- Peak estimation
- Distance estimation
- Reachable set estimation
- Maximum controlled invariant set
- Discounted infinite-time optimal control
- Time-delay analysis and control

Peak Estimation Problem

Find peak value P^* of $p(x)$ in state set X:

$$\begin{aligned} P^* = \sup_{t^*,\, x_0} \quad & p(x(t \mid x_0, w(\cdot)) \\ & \dot{x}(t) = f(t, x(t), w(t)) && \forall t \in [0, t^*] \\ & w(t) \in W(t, x(t)) && \forall t \in [0, t^*] \\ & x_0 \in X_0 \text{ (initial set)} \end{aligned}$$

Nonconvex problem in $(t^*, x_0, w(\cdot))$

Auxiliary Function

A function $v(t, x)$ that behaves nicely along trajectories

Examples:

- Value function
- Lyapunov function
- Barrier function

With w: nice along **all controlled trajectories**

Peak Function Program

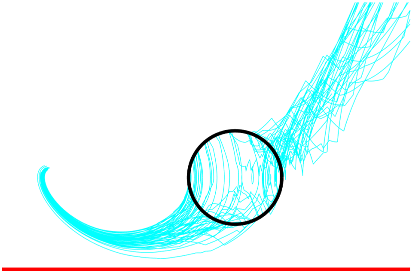
Infinite-dimensional LP[1] with auxiliary function $v(t,x)$

$$
\begin{aligned}
d^* = \inf_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(t,x) \geq p(x) && \forall (t,x) \in [0,T] \times X \\
& (\partial_t + f \cdot \nabla_x) v(t,x) \leq 0 && \forall (t,x,w) \in [0,T] \times X \times W(t,x) \\
& \gamma \geq v(0,x) && \forall x \in X_0 \\
& v \in C^1([0,T] \times X)
\end{aligned}
$$

$P^* = d^*$ if $[0,T] \times X$ compact, p l.s.c., f **Lipschitz**

[1]Cho, Moon Jung, and Richard H. Stockbridge. "Linear programming formulation for optimal stopping problems." SIAM Journal on Control and Optimization 40.6 (2002): 1965-1982.

System with Uncertainty Example



$$\dot{x}(t) = [x_2(t), -x_1 w(t) - x_2(t) + x_1(t)^3/3]$$

$$w(t) \in [0.5, 1.5]$$

Ways to Discretize

Infinite-dimensional LP must be discretized for computation

More complexity: more accurate solutions

Method	Increasing Complexity
Gridding (MDP)	# Grid Points
Basis Functions (ADP)	# Functions
Random Sampling	# Samples
Sum-of-Squares	Polynomial Degree
Neural Nets (FOSSIL)	Width and Depth
Your Favorite Method	Some Accuracy Parameter

Runtime usually exponential in dimension, complexity

Motivation and Size Comparison

PSD size w.r.t. states n, inputs L, dynamics degree $\tilde{d}$

$$\tilde{d} = \left\lceil \max_{\ell \in 0..L,\ v \in \mathbb{R}[t,x]_{\leq 2d}} (\deg f_\ell \cdot \nabla_x v)/2 \right\rceil$$

Maximal size of PSD matrices (SOS)

Size	Original	Robustified
Super	$\binom{n+L+\tilde{d}}{\tilde{d}}$	$\binom{n+\tilde{d}}{\tilde{d}}$

When $(n = 2, L = 10, d = 4, \tilde{d} = 6)$:

Original $= 8568$ (Mosek sad), Robust $= 56$ (Mosek happy)

Parameterized Robust Counterparts

Robust Counterpart Example: Box

Linear inequality constraint in β with robust uncertainty

Original β-feasible problem with unknown $\|w\|_\infty \leq 1$

$$\forall w : \quad a_0^T \beta + \sum_{\ell=1}^{L} w_\ell a_\ell^T \beta \leq b_0 + \sum_{\ell=1}^{L} w_\ell b_\ell \qquad (3)$$

Robust Counterpart Example: Box

Original β-feasible problem with unknown $\|w\|_\infty \leq 1$

$$\forall w : \quad a_0^T \beta + \sum_{\ell=1}^L w_\ell a_\ell^T \beta \leq b_0 + \sum_{\ell=1}^L w_\ell b_\ell \qquad (3)$$

Equivalent **Robust Counterpart** with w eliminated

$$\max_{\|w\|_\infty \leq 1} \left(\sum_{\ell=1}^L w_\ell [a_\ell^T \beta - b] \right) \leq b_0 - a_0^T \beta \qquad (4a)$$

$$\sum_{\ell=1}^L |a_\ell^T \beta - b| \leq b_0 - a_0^T \beta \qquad (4b)$$

Robust Counterpart Example: Box (cont.)

Express as finite-dimensional convex program

$$\sum_{\ell=1}^{L} \zeta_\ell^+ + \zeta_\ell^- \leq b_0 - a_0^T \beta \tag{5a}$$

$$\forall \ell : \qquad \zeta_\ell^\pm \geq \pm(a_\ell^T \beta - b_\ell) \tag{5b}$$

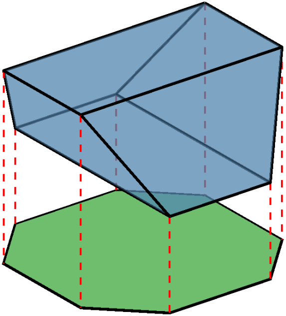
Optimization variables $(\beta, \zeta^\pm)$, $2L + 1$ constraints

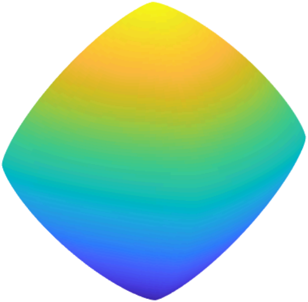
In this case, nonconservative to go infinite $\rightarrow$ finite program

Semidefinite-Representable (SDR) Set

Uncertainty $W \ni w$ (SDR with $K \subseteq \mathbb{S}^+$) [2]

$$W = \{w \in \mathbb{R}^L \mid \exists \lambda \in \mathbb{R}^\Lambda : Cw + G\lambda + e \geq_K 0\} \qquad (6)$$





[2] Fawzi, H., Gouveia, J., Parrilo, P. A., Saunderson, J., & Thomas, R. R. (2022). Lifting for simplicity: Concise descriptions of convex sets. SIAM Review, 64(4), 866-918.

Robust Counterparts (General)

Original problem with SDR uncertainty

$$\forall w \in W : \qquad a_0^T \beta + \sum_{\ell=1}^{L} w_\ell a_\ell^T \beta \le b_0 + \sum_{\ell=1}^{L} w_\ell b_\ell \quad (7)$$

Robust counterpart with dual variable ζ (sufficient)

$$e^T \zeta + a_0^T \beta \le b_0 \quad (8a)$$

$$G\zeta = 0,\ \zeta \ge_{K^*} 0 \quad (8b)$$

$$(C^T \zeta)_\ell + a_\ell^T \beta = b_\ell \qquad \forall \ell = 1..L \quad (8c)$$

Also necessary if (K convex, pointed, non-polyhedral Slater)[3]

[3] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, Robust Optimization. Princeton University Press, 2009, vol. 28

Robust Counterparts (Parameterized)

$(C, G, e, a_0, a_\ell, b_0, b_\ell)$ now depends on parameter $y \in P$

Strict robust inequality (with $\zeta(y)$ parameter-dependent):

$$\forall w \in W(y): \; a_0^T(y)\beta + \sum_{\ell=1}^{L} w_\ell a_\ell(y)^T \beta < b_0(y) + \sum_{\ell=1}^{L} w_\ell b_\ell(y)$$

Can we parameterize $\zeta(y)$ by continuous functions?

Conditions for Exactness

Necc. + Suff. conditions for exactness [4] [5]:

1. K convex, pointed, Slater if non-polyhedral
2. P is compact
3. All $(C(y), G(y), e(y), a_\bullet(y), b_\bullet(y))$ continuous in y
4. y-Slater: $\exists \zeta(y) >_{K^*} 0$ with $\forall y : [C(y)^T; G(y)^T]\zeta(y) = 0$

[4] Miller, Jared, and Mario Sznaier. "Analysis and Control of Input-Affine Dynamical Systems using Infinite-Dimensional Robust Counterparts," arXiv:2112.14838, 2023.

[5] Conditions based on: J. Denel, "Extensions of the continuity of point-to-set maps: applications to fixed point algorithms," Point-to-Set Maps and Mathematical Programming, pp. 48–68, 1979

Polynomial Approximability

Nonconservative to Stone-Weierstrass approximate by continuous functions?

So far: $(C(y), G(y))$ are constant in y

Open question: less stringent conditions on $(C(y), G(y))$?

Revisiting Peak Estimation

Robust counterpart with $W(t, x) = \{w \mid Aw \leq b\}$

$$
\begin{aligned}
d^* = \ & \min_{\gamma \in \mathbb{R}} \gamma \\
& \gamma \geq v(0, x) & & \forall x \in X_0 \\
& \mathcal{L}_{f_0} v(t, x) + b^T \zeta(t, x) \leq 0 & & \forall (t, x) \in [0, T] \times X \\
& (A^T)_\ell \zeta(t, x) = (f_\ell \cdot \nabla_x) v(t, x) & & \forall \ell = 1..L \\
& v(t, x) \geq p(x) & & \forall (t, x) \in [0, T] \times X \\
& v(t, x) \in C^1([0, T] \times X) & & \\
& \zeta_k(t, x) \in C_+([0, T] \times X) & & \forall k = 1..m
\end{aligned}
$$

Only the Lie-derivative dynamics constraint changes

Optimal Control Problems

Optimal Control Setup

Standard optimal control problem (Bolza-form)

$$
\begin{aligned}
P^* = \inf_{u(t)} \int_{t=0}^{T} J(t, x(t), u(t))dt + J_T(x(T)) & \\
\dot{x}(t) = f(t, x(t), u(t)),\ \ u(t) \in U & \qquad \forall t \in [0, T] \\
x(0) = x_0,\ \ x(T) \in X_T &
\end{aligned}
$$

Assume input-affine $f(t, x, u) = f_0(t, x) + \sum_{j=1}^{m} u_j f_j(t, x)$

Optimal Control Program

HJB Equations for Value Function V^*

$$0 = J_T(x(T)) - V^*(T, x(T))$$
$$0 = \inf_{u \in U} \left(\dot{V}^*(t, x(t), u) + J(t, x(t), u(t)) \right) \quad \forall t \in [0, T]$$

HJB Inequalities for subvalue $v(t, x) \leq V^*(t, x)$

$$J_T(x) - v(T, x) \geq 0 \quad \forall x \in X_T$$
$$J(t, x, u) + \dot{v}(t, x, u) \geq 0 \quad \forall (t, x, u) \in [0, T] \times X \times U$$

Optimal Control Relaxation

Tightest subvalue at initial condition x_0

$$
\begin{aligned}
p^*(x_0) = \sup_v \; & v(0, x_0) \\
& J_T(x) - v(T, x) \geq 0 && \forall x \in X_T \\
& J(t, x, u) + \dot{v}(t, x, u) \geq 0 && \forall (t, x, u) \in [0, T] \times X \times U \\
& v \in C^1([0, T] \times X)
\end{aligned}
$$

$p^*(x_0) = V^*(0, x_0)$ if (compact, regularity, convexity) [6]

[6] Lewis, Richard M., and Richard B. Vinter. "Relaxation of optimal control problems to equivalent convex programs." Journal of Mathematical Analysis and Applications 74.2 (1980): 475-493.

Robustified Lie Constraint

Assuming that U, $\mathrm{Epigraph}(J)$ are (t, x)-SDR, define

$$\bar{U}(t, x) = \{u \in U(t, x),\ \tau \in [J(t, x, u), \max_{t,x,u}(J)]\} \tag{9}$$

Epigraph-expanded Lie constraint (includes LQR):

$$\begin{aligned} &\forall (t, x) \in [0, T] \times X,\ \forall (\tau, u) \in \bar{U}(t, x) : \\ &\quad \tau + (\partial_t + f_0 \cdot \nabla) v + \textstyle\sum_{j=1}^{m} u_j f_j \cdot \nabla_x v \geq 0 \end{aligned} \tag{10}$$

Robustify to eliminate (u, τ), $(1 + n)$ variables left (t, x)

Prior Robustified Dynamics

Parameterized and robustified dynamical systems problems

- L_1-optimal OCP with $U = \text{Box}$[7] [8]
- Density Functions with $U = \text{Polytope}$ ($X = \mathbb{R}^n$: Suff.) [9]

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[7] A. Majumdar, R. Vasudevan, M. M. Tobenkin, and R. Tedrake, "Convex Optimization of Nonlinear Feedback Controllers via Occupation Measures," The International Journal of Robotics Research, 2014

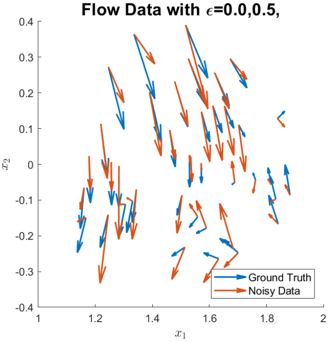
[8] M. Korda, D. Henrion, and C. N. Jones, "Controller design and value function approximation for nonlinear dynamical systems", Automatica, 2016

[9] Dai, Tianyu, and Mario Sznaier. "A semi-algebraic optimization approach to data-driven control of continuous-time nonlinear systems." IEEE L-CSS, 2020.

Data-Driven Analysis

Sampling: Flow System

Data $\mathcal{D} = \{(t_j, x_j, \dot{x}_j)\}_j$ under mixed L_∞-bounded noise



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Dynamics Model

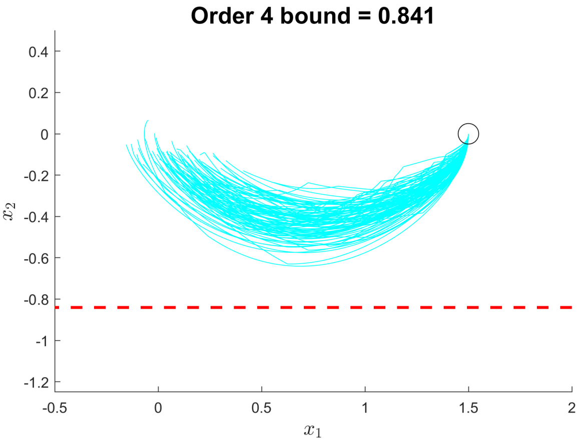
Given data $\mathcal{D}$, budget ϵ, system model $\{f_0, f_\ell\}$

Parameterize ground truth F by functions in dictionary

$$\dot{x}(t) = f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x)$$

Treat unknown parameters w as adverse inputs

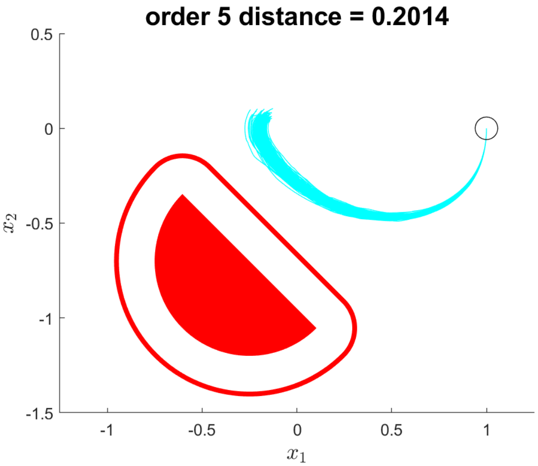
Peak Estimation Example (Flow)



$$\dot{x} = [x_2, \; \text{cubic}(x_1, x_2)]$$

$$L = 10, \; m = 80 \text{ (33 nonredundant)}$$

Distance Estimation Example (Flow)

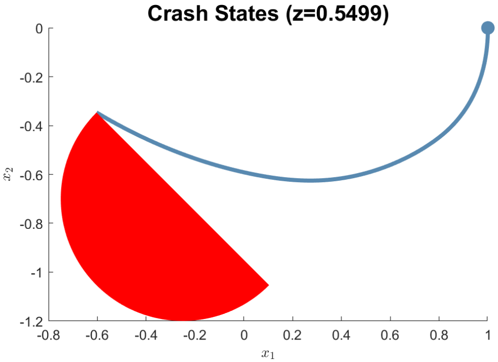


$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

$$L = 10,\ m = 80\ (33\ \text{nonredundant})$$

Crash-Bound Estimation (Flow)

Minimum L_∞ data corruption needed to crash into unsafe set[10]

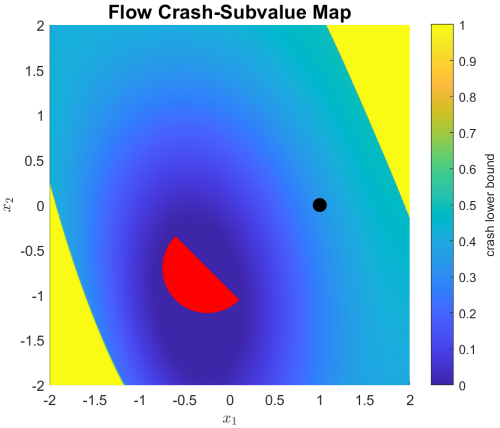


True $\epsilon = 0.5$, distance ≈ 0.2014

[10] Jared Miller and Mario Sznaier. "Quantifying the Safety of Trajectories using Peak-Minimizing Control", arXiv:2303.11896, 2023.

Crash-Subvalue (Flow)

Subvalue map: lower-bound on ϵ needed to crash



Bound at • of $0.3399 \leq 0.5499$, but valid everywhere in X

Discrete-Time OCP/Peak

Lack of Linearity

Discrete-time dynamics $x_+ = f(x, u) = f_0(x) + \sum_j u_j f_j(x)$

If $v(x)$ nonlinear, then $v(f(x, u))$ nonlinear in u

$$\forall (x, u) \in X \times U$$
$$J(x, u) + \left(v(x) - v\left(f_0(x, u) + \sum_{j=1}^m u_j f_j(x, u) \right) \right) \geq 0.$$

Obstacle to discrete-time optimal control [11]

[11] W. Han and R. Tedrake, "Controller Synthesis for Discrete-Time Polynomial Systems via Occupation Measures," IEEE IROS, 2018

Linear with expanded state

New variable $\tilde{x}$ with support:

$$\Omega = \{(x, \tilde{x}, u) \in X^2 \times U \mid \tilde{x} = f(x, u)\}$$

Equivalent HJB inequality:

$$J(x, u) + v(x) - v(\tilde{x}) \geq 0 \qquad \forall (x, \tilde{x}, w) \in \Omega.$$

Robustify to eliminate u, left with $(t, x, \tilde{x})$

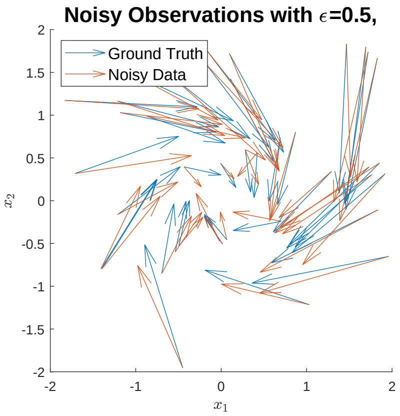
Unlocked Possibilities

Favorable when $m > n$, substitute $(t, x, u) \to (t, x, \tilde{x})$

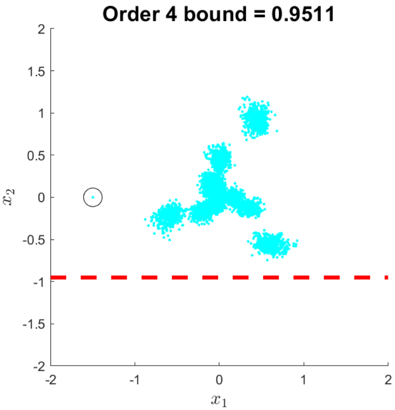
Optimal control and data-driven analysis of:

- Discrete-time systems
- Hybrid systems (resets)
- Stochastic analogues (jump maps)

Peak Estimation in Discrete-Time



(a) Observed data of



(b) Order 4 tightening

$$\text{Quadratic Model of } x_+ = \begin{bmatrix} -0.3x_1 + 0.8x_2 + 0.1x_1x_2 \\ -0.75x_1 - 0.3x_2 \end{bmatrix}$$

Take-aways

Conclusion

Input affine structure is powerful and expensive

Robustify to reduce complexity

Applicable to many dynamical systems problems

Thank you for your attention

Questions?

Bonus (Sum-of-Squares)

Sum-of-Squares Method

Every $c \in \mathbb{R}$ satisfies $c^2 \geq 0$

Sufficient: $q(x) \in \mathbb{R}[x]$ nonnegative if $q(x) = \sum_i q_i^2(x)$

Exists $v(x) \in \mathbb{R}[x]^s$, *Gram* matrix $Z \in \mathbb{S}_+^s$ with $q = v^T Z v$

Sum-of-Squares (SOS) cone $\Sigma[x]$

$$\begin{aligned} & x^2y^4 - 6x^2y^2 + 10x^2 + 2xy^2 + 4xy - 6x + 4y^2 + 1 \\ =& (x + 2y)^2 + (3x - 1 - xy^2)^2 \end{aligned}$$

Motzkin Counterexample (nonnegative but not SOS)

$$x^2y^4 + x^4y^2 - x^2y^2 + 1$$

Sum-of-Squares Method (cont.)

Putinar Positivestellensatz (Psatz) nonnegativity certificate over set $\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}$ ($\Sigma[\mathbb{K}]$:

$$q(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x)$$
$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma_i(x) \in \Sigma[x], \quad \phi_j \in \mathbb{R}[x]$$

Psatz at degree $2d$ is an SDP, monomial basis: $s = \binom{n+d}{d}$

Archimedean: $\exists R \geq 0$ where $R - \|x\|_2^2$ has Psatz over $\mathbb{K}$

$\mathbb{K}$ Archimedean: all $\mathbb{K}$-positive polynomials are in $\Sigma[\mathbb{K}]$

Polynomial Matrix Inequalities

SOS method (scalar): $q(x) \geq 0$

Extend to matrices $Q(x) \in \mathbb{S}^s_{++}$

SOS matrix: $Q(x) = R(x)^T R(x) \in \Sigma^s[x]$ for matrix $R(x)$

Gram matrix (PSD) constraint of size $s\binom{n+d}{d}$

Scherer Psatz: nonnegativity over constraint sets

Bonus (Preprocessing)

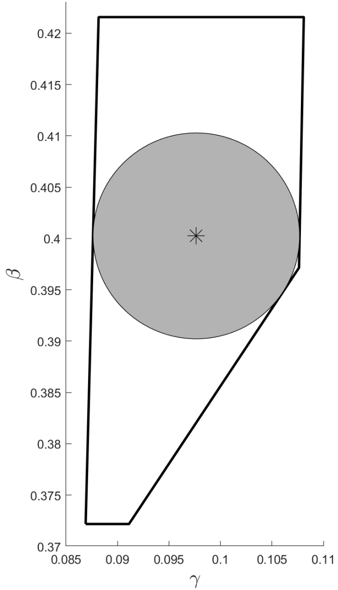
Preprocessing: Centering

Chebyshev center c: center of sphere with largest radius in W

Find through linear programming

$$\max r$$
$$A_k c + r\|A_k\|_2 \leq b_k \qquad \forall k$$
$$r \geq 0, c \in \mathbb{R}^L$$

Shifted dynamics $f_0 \leftarrow f_0 + \sum_{\ell=1}^{L} c_\ell f_\ell$

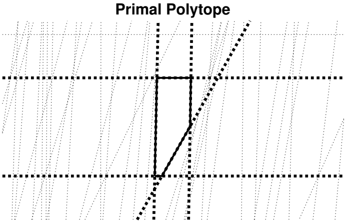


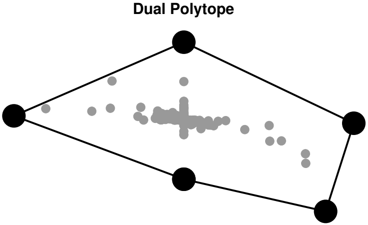
Preprocessing: Redundancy

Majority of $m = 2N_x N_s$ constraints are often redundant

Convex hull of dual polytope:
Time: $\Omega(m \log m + m^{\lfloor L/2 \rfloor})$

Linear program per constraint:
Time: $m \times \tilde{O}(mL + L^3)$ [a]
(Jan van den Brand *et. al.* 2020)





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[a] van den Brand, Jan, "A deterministic linear program solver in current matrix multiplication time". Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 2020.

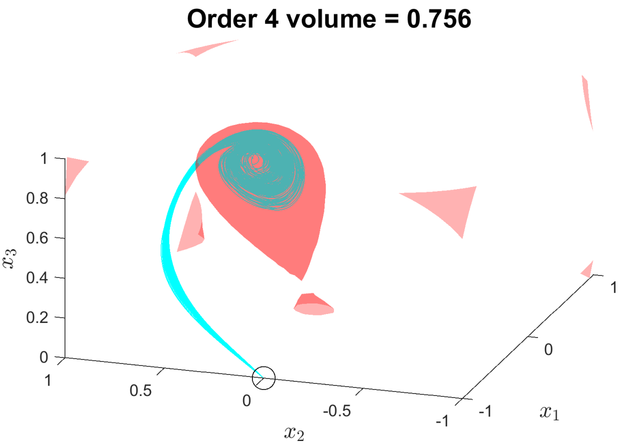
Bonus (OCP)

Assumptions

1. U and Graph(J) are (t, x)-SDR
2. $[0, T] \times X \times U$ is compact
3. Input-affine dynamics $f = f_0 + \sum_{j=1}^{m} u_j f_j$
4. Each f_0, f_ℓ is continuous
5. Original HJB inequality relaxation is tight

Assumption 1: L_1-optimal, L_∞-optimal, LQR, etc.

Reachable Set Estimation Example (Twist)



Unknown A, Known B

$L = 9, \ m = 600$ (34 nonredundant)

Bonus (Crash-Safety)

Crash-Safety

Corruption in L_∞-bounded setting

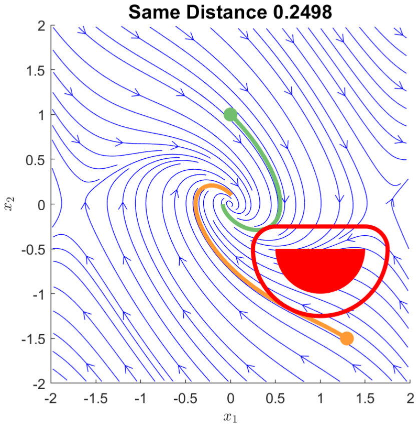
$$J(w) = \max_k \| f_0(t_k, x_k) + \textstyle\sum_{\ell=1}^{L} w_\ell f_\ell(t_k, x_k) - y_k \|_\infty$$
$$= \max(h - \Gamma w) \quad \text{for some polytope } \ (\Gamma, h)$$

How much data corruption is needed to crash?

$$Q^* = \inf_{t,\ x_0,\ w} \ \sup_{t' \in [0,t]} J(w(t'))$$
$$\dot{x}(t') = f(t', x(t'), w(t')) \qquad \forall t' \in [0, T]$$
$$x(t \mid x_0, w(\cdot)) \in X_u$$
$$w(\cdot) \in W,\ \ t \in [0, T],\ \ x_0 \in X_0$$

Example Crash-Bounds

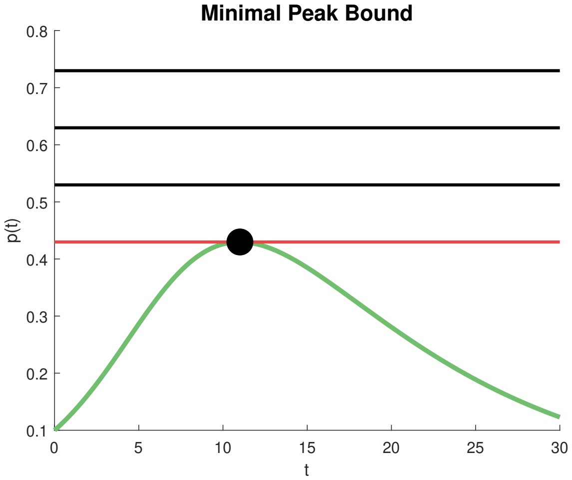
Two trajectories have same distance, different crash-bounds



Green-Top $Q^* = 0.316$, Yellow-Bottom $Q^* = 0.622$

Peak Minimizing Control

Find minimum bound on the maximum p value



Crash-safety is Peak Minimizing Control

Peak-Minimizing Control

Add state $\dot{z} = 0$ (Molina, Rapaport, Ramírez 2022)

$$\begin{aligned}
Q_z^* = \inf_{t,\ x_0,\ z,\ w} \ & z \\
& \dot{x}(t') = f(t', x(t'), w(t')) & & \forall t' \in [0, T] \\
& \dot{z}(t') = 0 & & \forall t' \in [0, T] \\
& J(w(t')) \leq z & & \forall t' \in [0, T] \\
& x(t \mid x_0, w(\cdot)) \in X_u \\
& w(\cdot) \in W, \ \ t \in [0, T] \\
& x_0 \in X_0, z \in [0, J_{\max}]
\end{aligned}$$

Drive down the z-upper-bound on $J(w)$

Crash-Bound Program

Consistency sets

$$Z = [0, J_{\max}] \qquad \Omega = \{(w, z) \in W \times Z : J(w) \leq z\}.$$

Optimal Control Problem with auxiliary $v(t, x, z) \in C^1$

$$\begin{aligned} d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\ & v(0, x, z) \geq \gamma && \forall (x, z) \in X_0 \times Z \\ & v(t, x, z) \leq z && \forall (t, x, z) \in [0, T] \times X_u \times Z \\ & \mathcal{L}_f v(t, x, z, w) \geq 0 && \forall (t, x, z, w) \in [0, T] \times X \times \Omega \end{aligned}$$

Crash Lie-decomposition

Exploit affine structure of $J(w) = \max_j (h - \Gamma w)_j$

Nonconservatively robustified Lie constraint

$$
\begin{aligned}
d^* = \sup_{\gamma \in \mathbb{R},\ v} \quad & \gamma \\
& v(0, x, z) \geq \gamma & & \forall (x, z) \in X_0 \times Z \\
& v(t, x, z) \leq z & & \forall (t, x, z) \in [0, T] \times X_u \times Z \\
& \mathcal{L}_{f_0} v - (z\mathbf{1} + h)^T \zeta \geq 0 & & \forall (t, x, z) \in [0, T] \times X \times [0, J_{\max}] \\
& (\Gamma^T)_\ell \zeta + f_\ell \cdot \nabla_x v = 0 & & \forall \ell = 1..L \\
& \zeta_j \in C_+([0, T] \times X \times Z) & & \forall j = 1..2nT
\end{aligned}
$$