Decomposed Structured Subsets for Semidefinite Optimization

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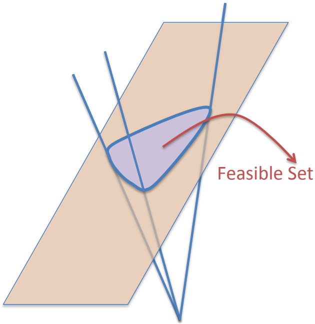
Semidefinite Programs

Modern control problems require solutions of large scale SDPs

$$\min_X \langle C, X \rangle$$

$$b_i = \langle A_i, X \rangle \,|_{i=1}^{m}$$

$$X \in \mathbb{S}_+^n$$

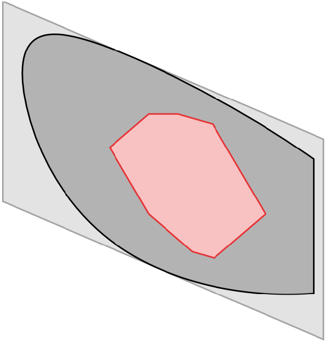


Scales polynomially with n and m

Approximations of SDPs

Approximate large SDPs

Structured (simpler) subset of PSD



$\mathcal{DD} \subset \mathbb{S}_+ \subset \mathcal{DD}^*$

Diagonal Dominance ($\mathcal{DD}$):

$$X_{ii} \geq \Sigma_{i \neq j} |X_{ij}|$$

bounds by Linear Programming

Generalizes to Factor Width

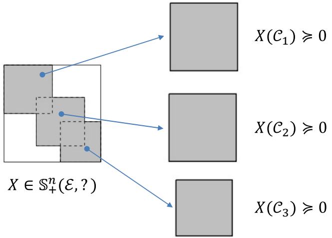
Structure in SDPs

Use structure to simplify problem

- Sparsity
- Symmetry
- Algebraic

Large PSD cone $\to$ smaller cones

Often renders problems tractable



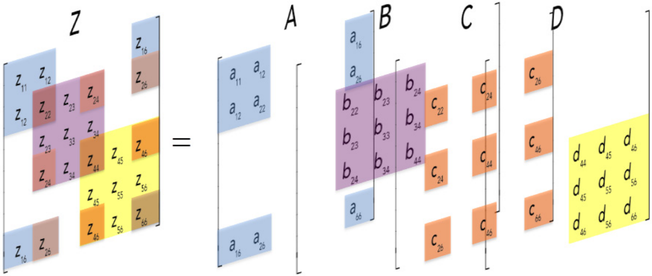
Blocks agree on overlap

Sparse PSD Matrices

Example of Chordal Sparsity

$$Z \succeq 0 \quad \Leftrightarrow \quad \exists\, A,\ B,\ C,\ D \text{ such that}$$

$$A + B + C + D = Z, \qquad A,\ B,\ C,\ D \succeq 0$$



Large sparse PSD $\leftrightarrow$ sum of smaller PSD

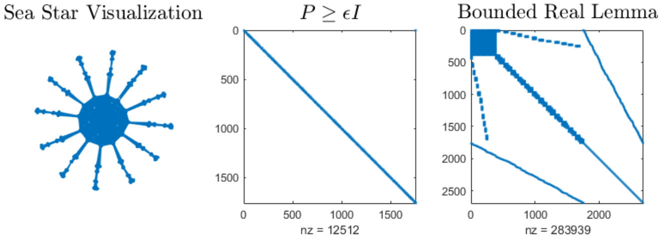
Network H-infinity Estimation

Small Gain theorem: G robustly stable if $\|G\|_\infty < 1$

Bounded Real Lemma: $\|G\|_\infty < \gamma$ iff $\exists P \succ 0$

$$\begin{bmatrix} PA + A^\mathsf{T}P + C^\mathsf{T}C & P^\mathsf{T}B + C^\mathsf{T}D \\ B^\mathsf{T}P + D^\mathsf{T}C & -\gamma^2 I \end{bmatrix} \prec 0.$$

Block-diagonal P inherits sparsity of network



Sea Star network topology and LMI sparsity

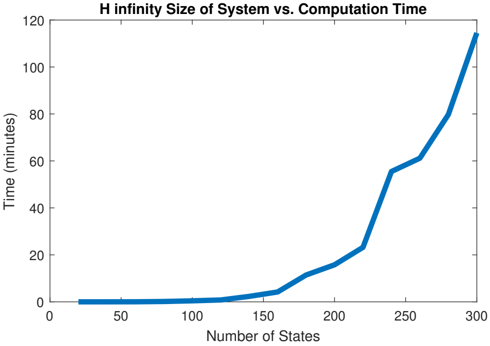
Sea Star

Inspiration for Sea Star Network

H-infinity Optimization

No structure: $\approx$ 2 hours for 300 states

Out of memory: $\geq$ 400 states (64 GB RAM)



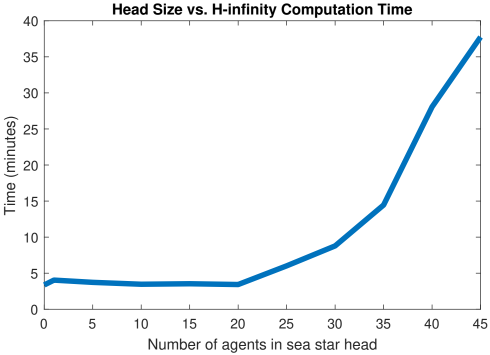
Dense H_∞ optimization

H-infinity on Sea Star

Sea Star structure with chordal decomposition

Head sizes $0 \ldots 45$, largest clique 242

Out of memory: Head Size ≥ 50



Motivating Example M(a,b)

For what values of (a, b) is $M(a, b)$ PSD-completable?

$$M(a,b) = \begin{bmatrix} 1 & \frac{1}{2}+a & ? & ? \\ \frac{1}{2}+a & 2-b & -2a & a+b \\ ? & -2a & 5 & b/2 \\ ? & a+b & b/2 & 2 \end{bmatrix}$$

Cliques are $\mathcal{C}_1 = (1, 2) \qquad \mathcal{C}_2 = (2, 3, 4)$

Decomposed Structured Subsets

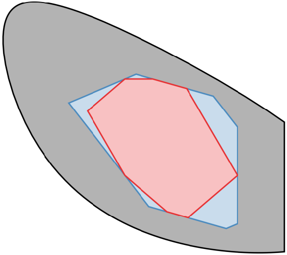
Structured Subsets destroy problem properties

Decompose, then approximate

Blocks are $\mathcal{DD}$ vs. Matrix is $\mathcal{DD}$

$$\begin{pmatrix} X_{11} & X_{12} & ? & ? \\ X_{12} & X_{22} & X_{23} & X_{24} \\ ? & X_{23} & X_{33} & X_{34} \\ ? & X_{24} & X_{34} & X_{44} \end{pmatrix} \in \mathbb{S}^4_+$$

Cliques: $\{(1,2),(2,3,4)\}$

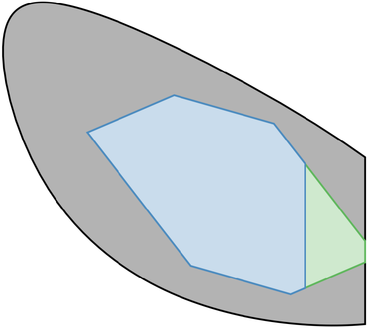


$\mathcal{DD} \subset \{\mathcal{DD}, \mathcal{DD}\} \subset \mathbb{S}_+$

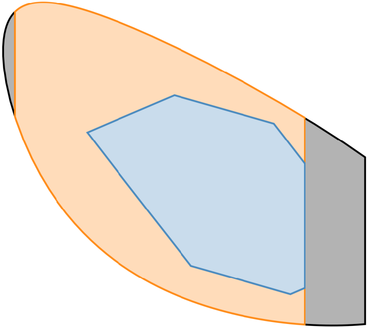
Preserving structure yields tighter approximations

Mixing Cones

Adds flexibility in optimization



$\{\mathbb{S}_+, \mathcal{DD}\}$

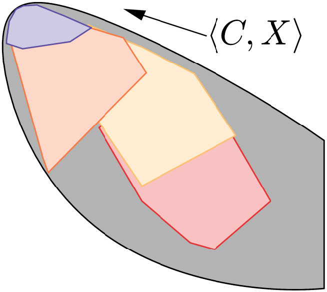


$\{\mathcal{DD}, \mathbb{S}_+\}$

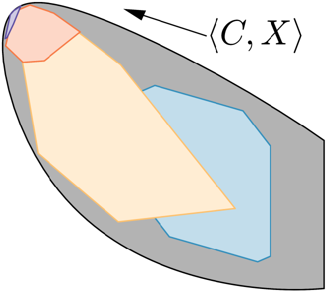
Useful if problem has few large cliques, bottleneck

Change of Basis

Iteratively refine approximations (Ahmadi, Hall 2015)



Start from $\mathcal{DD}$



Start from $\{\mathcal{DD}, \mathcal{DD}\}$

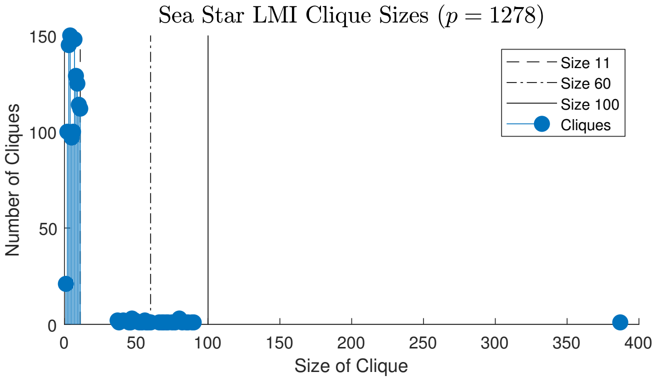
Standard vs. Decomposed Change of Basis

Improved solution quality with structure

Sea Star Cliques

310 agents, each a random linear system of order ≤ 10

70 in 'head', 20 in each of the 12 'arms' in 2 'knuckles'

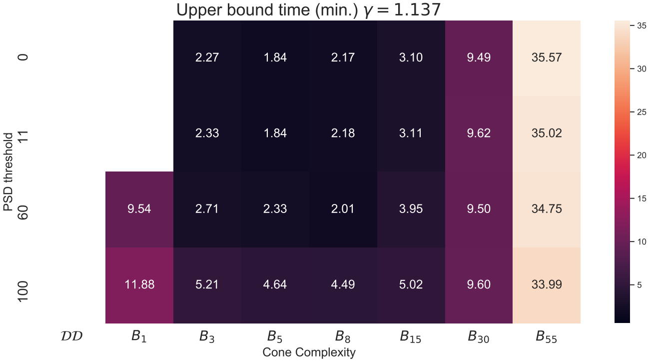


Single clique sized 387, rest have size < 90

Sea Star Upper Bounds

Cliques larger than threshold remain PSD

B_k: Block-factor-width 2 matrices, k-sized blocks

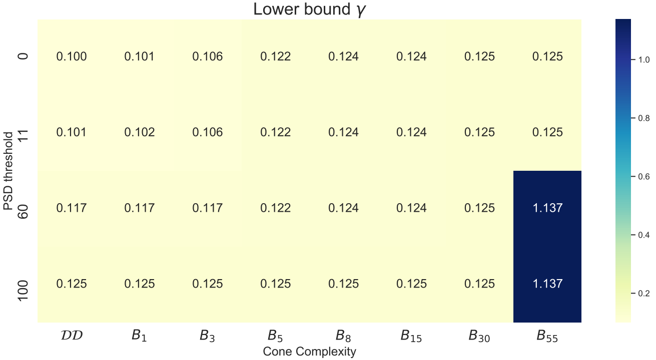


Time to find γ by upper bound K (minutes)

Sea Star Lower Bounds

Dual decomposed structured subsets for lower bounds

Can verify if approximation = SDP solution (KKT)



γ found by lower bound K^*

Conclusions

Structure in SDPs is powerful

Decompositions allow for tractable programs

Preserve structure while approximating

Increased performance and tighter bounds with structure

https://github.com/zhengy09/SDPfw