Data-Driven Superstabilizing Control under Quadratically-Bounded Errors-in-Variables Noise

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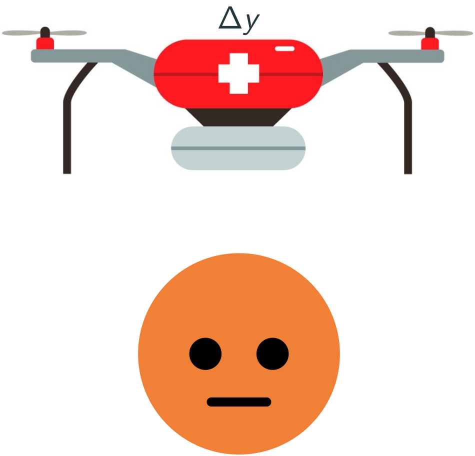
Motivating Scenario

Flying a drone

Flying a drone

Flying a drone



Flying a drone

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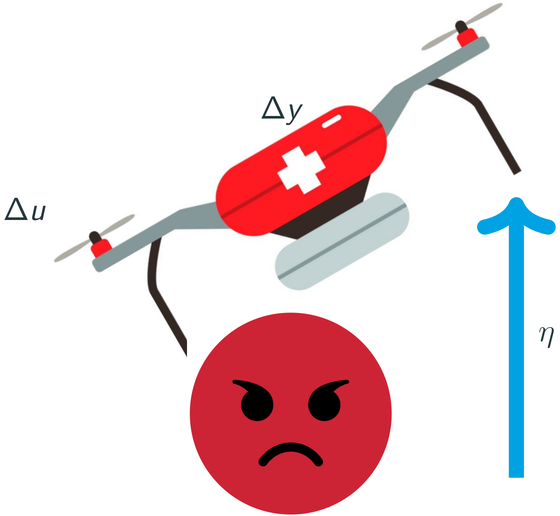
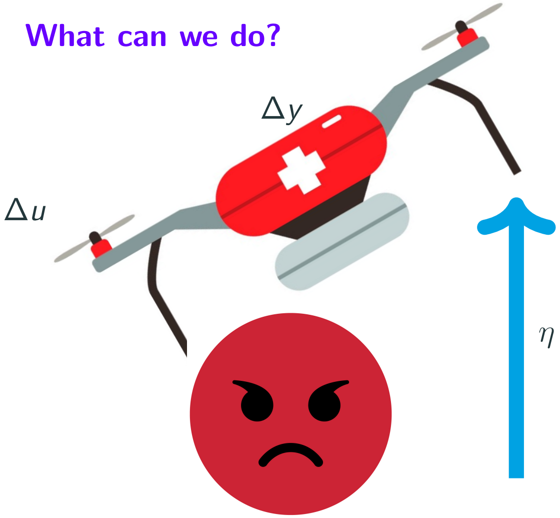


Flying a drone

What can we do?



Errors-in-Variable Noise Task

Noisy measurements $\mathcal{D} = \{\hat{x}_t, \hat{u}_t\}_{t=1}^{T}$ of linear system ($y = x$)

$$x_{t+1} = Ax_t + Bu_t + E\eta_t \tag{1a}$$

$$\hat{x}_t = x_t + \Delta x_t \tag{1b}$$

$$\hat{u}_t = u_t + \Delta u_t \tag{1c}$$

Errors-in-Variables (EIV): $\Delta x, \Delta u \neq 0$

Want to control system under EIV noise

Bilinear Trouble

$(A, B, \Delta x, \Delta u, \eta)$ all unknown

$$\hat{x}_{t+1} - \Delta x_{t+1} = A\hat{x}_t - A\Delta x_t + Bu_t - B\Delta u_t - E\eta_t$$

Multiplication between unknown $A\Delta x_t$, also in $B\Delta u_t$

Nonconvex identification, control problems [1]

[1] T. Söderström. Errors-in-Variables Methods in System Identification. Springer, 2018

What can we do for EIV control?

Behavioral: (soft noise bounds)

DeepC: Regularize against errors-in-variables [2]

Set Membership: (hard noise bounds)

Quadratic Matrix Inequalities, S-Lemma[3]

Polyhedral Lyapunov Functions and Robust Counterparts[4] [5]

[2] I. Markovsky, L. Huang, F. Dörfler. Data-driven control based on the behavioral approach: From theory to applications in power systems. IEEE Control Systems Magazine, 2023

[3] Lidong Li, Andrea Bisoffi, Claudio De Persis, Nima Monshizadeh. Controller synthesis from noisy-input noisy-output data. In IEEE CDC, 2024

[4] Jared Miller, Tianyu Dai, and Mario Sznaier. Data-Driven Superstabilizing Control of Error-in-Variables Discrete-Time Linear Systems. In IEEE CDC, 2022

[5] This presentation that you are attending

EIV Sets and Control Problem

Quadratic Noise Sets

Noise described by L **quadratic** constraints

$$\forall \ell \in 1..L : \quad \|F_\ell \Delta x + G_\ell \Delta u\|_2 \leq 1, \tag{2}$$

Second-order-cone ($\mathbb{L}$) equivalent description

$$\forall \ell (F_\ell \Delta x + G_\ell \Delta u, 1) \in \mathbb{L}^{n_\ell} \tag{3}$$

- elementwise norm constraints
- energy bounds
- subgaussian chance constraints

Consistency Set

Consistency set $\bar{\mathcal{P}}(A, B, \Delta x, \Delta u)$ (with $\eta = 0$)[6]

$$\bar{\mathcal{P}} := \begin{cases} \Delta x_{t+1} = A\Delta x_t + B\Delta u_t + h_t^0 & \forall t \in 1..T-1 \\ \|F_\ell \Delta x + G_\ell \Delta u\|_2 \leq 1 & \forall \ell \in 1..L \end{cases}, \quad (4)$$

Affine weight h^0 (residual) is defined by

$$h_t^0 = \hat{x}_{t+1} - A\hat{x}_t - Bu_t \qquad \forall t = 1..T-1$$

[6] V. Cerone. Feasible parameter set for linear models with bounded errors in all variables. Automatica, 29(6): 1551–1555, 1993.

Polyhedral Lyapunov Function Definition

System $x_+ = (A + BK)x$ is **superstable**[7,8] if:

- $\|Wx\|_\infty$ is a Lyapunov Function
- $\|W(A + BK)W^+\|_\infty < 1$

Performance bounds on decay:

$$\|W(A + BK)W^+\|_\infty = \gamma \quad \Longrightarrow \quad \|Wx_t\|_\infty \leq \gamma^t \|Wx_0\|_\infty$$

[7] F. Blanchini and M. Sznaier. Persistent disturbance rejection via static-state feedback. IEEE Transactions on Automatic Control

[8] B. T. Polyak and P. S. Shcherbakov. Superstable Linear Control Systems. I. Analysis. Automation and Remote Control, 63(8):1239–1254, 2002

Stabilization problem for EIV

PLFs are universal for switched systems

But finding a PLF is undecidable

Control Problem (for fixed $W \in \mathbb{R}^{f \times n}$):

Find $K \in \mathbb{R}^{m \times n}$ s.t. $\|W(A + BK)W^{+}\|_{\infty} < 1 \; \forall (A, B) \in \mathcal{P}$

Stability Application

LP with $2f^2 + f$ inequalities over $(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}}$

$$\text{find}_{M,K} \ \text{s.t.} \ \forall (A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}} : \tag{5a}$$

$$\forall i = 1..f : \tag{5b}$$

$$1 - \eta - \textstyle\sum_{j=1}^{n} M_{ij}(A, B) \geq 0$$

$$\forall i = 1..f, \ j = 1..f : \tag{5c}$$

$$M_{ij}(A, B) - (W(A + BK) W^{+})_{ij} \geq 0$$

$$M_{ij}(A, B) + (W(A + BK) W^{+})_{ij} \geq 0$$

$$M(A, B) : \mathcal{P} \to \mathbb{R}^{f \times f}, \ K \in \mathbb{R}^{m \times n}. \tag{5d}$$

Nonconservative if $\bar{\mathcal{P}}$ compact (M continuous map)

How do we solve infinite LPs?

Discretization necessary to solve on computer

More complexity: more accurate solutions

Method	**Increasing Complexity**
Gridding	# Grid Points
Basis Functions	# Functions
Random Sampling	# Samples
⋆ Sum-of-Squares (SOS)	Polynomial Degree
Your Favorite Method	Some Accuracy Parameter

Runtime usually exponential in dimension, complexity

Scaling of SOS Matrix Programs

Is a matrix-valued-polynomial $Q \in \mathbb{S}^s[x]$ positive definite?

SOS matrix: $Q(x) = R(x)^T R(x)$ for matrix $R(x) \in \mathbb{R}^{\ell \times s}$

N variables, degree k, PSD matrix size $s\binom{N+k}{k}$

Hol-Scherer Psatz: nonnegativity over constraint sets

Sufficient certificate, necessary if archimedean condition holds

Size Impact: Full

Without robustification, enumeration over $(A, B, \Delta x, \Delta u)$

$$p_F = \binom{n(n+T) + m(n+T-1) + k}{k}.$$

Absolutely awful scaling, drowns Mosek with low k

Reduced Scaling: Robust Counterparts

Eliminate $\Delta x, \Delta u$ to reduce dimensionality

Second-order-cone ($\mathbb{L}$) description of $\bar{\mathcal{P}}(\Delta x, \Delta u; A, B)$

$$\Xi := [(I_{T-1} \otimes A), \mathbf{0}_{n(T-1)\times n}] + [\mathbf{0}_{n(T-1)\times n}, -I_{n(T-1)}], \tag{6}$$

$$\bar{\mathcal{P}} := \left\{ \begin{matrix} \Xi\Delta x + (I_{T-1} \otimes B)\Delta u + h^0 = 0 & \\ (F_\ell \Delta x + G_\ell \Delta u, 1) \in \mathbb{L}^{n_\ell} & \forall \ell \in 1..L \end{matrix} \right\}. \tag{7}$$

$\bar{\mathcal{P}}$ is an ellipsoid in $\Delta x, \Delta u$ for fixed (A, B)

Equivalence of Statements

Each l.h.s. expression is independent of (A, B)

$$q(A, B) \geq 0 \qquad \forall (A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}} \tag{8}$$

Equivalent to robust counterpart with dual variables (μ, s, τ)

$$\text{find}_{\mu, s, \tau} \ \ \text{s.t.} \ \forall (A, B) \in \mathcal{P} : \tag{9a}$$

$$q - \textstyle\sum_{\ell=1}^{L} \tau_\ell - \mu^\top h^0 \geq 0 \tag{9b}$$

$$\Xi^\top \mu - \textstyle\sum_{\ell=1}^{L} F_\ell^\top s_\ell = 0 \tag{9c}$$

$$(I_{T-1} B^\top) \mu - \textstyle\sum_{\ell=1}^{L} G_\ell^\top s_\ell = 0 \tag{9d}$$

$$\mu : \mathcal{P} \to \mathbb{R}^{n(T-1)} \tag{9e}$$

$$(s_\ell, \tau_\ell) : \mathcal{P} \to \mathbb{L}^{n_\ell} \qquad \forall \ell \in 1..L. \tag{9f}$$

Second-Order Cone realizations

How to implement $(s_\ell, \tau_\ell) \in \mathbb{L}^{n_\ell} \forall \ell \in 1..L$?

Dense (block-arrow)

$$\begin{bmatrix} \tau_\ell(A,B) & s_\ell(A,B) \\ s_\ell(A,B) & \tau_\ell(A,B) I_n \end{bmatrix} \in \text{SOS Matrix}^{n_\ell+1} \qquad (10)$$

Sparse (small blocks)

$$\exists z_\ell : \quad \begin{bmatrix} \sum_{i=1}^{n_\ell} z_{i\ell}(A,B) & s_{i\ell}(A,B) \\ s_{i\ell}(A,B) & z_{i\ell}(A,B) \end{bmatrix} \in \text{SOS Matrix}^2 \qquad (11)$$

Sum of Squares Comparison

Sizes of PSD matrices

No robustification

$$p_F = \binom{n(n+T) + m(n+T-1) + k}{k}.$$

Dense: $(n_\ell + 1 \times n_\ell + 1)$ SOS matrix

$$p_A^\ell = (n^\ell + 1)\binom{n(n+m) + k}{k}.$$

Sparse: n_ℓ instances of (2×2) SOS matrices, adds conservatism

$$p_B = 2\binom{n(n+m) + k}{k}.$$

Comparison of Conservatism

This work

- Polytopic Lyapunov Function
- Choice of matrix W
- Polynomial degree
- Sparse formulation

QMI work [9]

- Signal to Noise Ratio
- Inconsistent noise values between time steps

[9] A. Bisoffi, L. Li, C. D. Persis and N. Monshizadeh, "Controller Synthesis for Input-State Data With Measurement Errors," in IEEE Control Systems Letters, vol. 8, pp. 1571-1576, 2024

Extensions

Extended Superstability: design diagonal W matrix

Suboptimal (conservative) H_2 or H_∞ control

Other uncertainty sets

Take-aways

Conclusion

Stabilization in the Error-in-variables setting

Robustification to reduce complexity

Thank you for your attention

Questions?