Data-Driven Control under Input and Measurement Noise

Jared Miller
Tianyu Dai
Mario Sznaier

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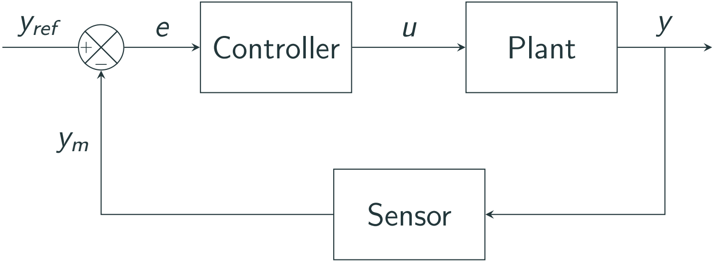
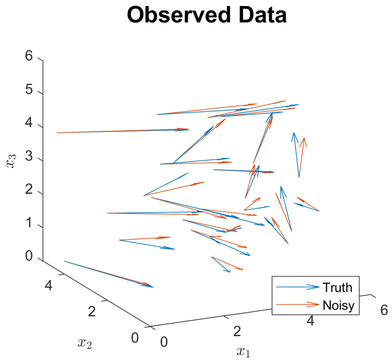
What is Data-Driven Control?

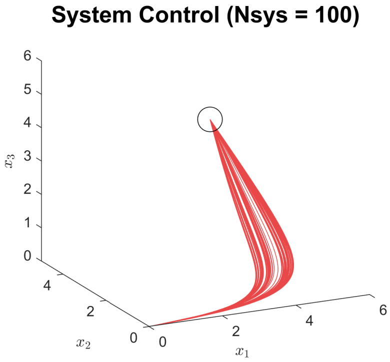
Design a controller for an unknown plant



Control system directly from data, no sysid required

Example of Data-Driven Control





Single controller stabilizes all data-consistent plants

Algorithms for Data-Driven Control

Virtual Reference Feedback Tuning (first methods)

Set-Membership (this talk)

- (Data-consistent plants) $\subseteq$ (K-Stabilized plants)
- Certificates: Farkas, Interval, S-Lemma, SOS

Behavioral

- Parameterize and pick out best system trajectory (MPC)
- Willem's Fundamental Lemma (DeePC)

Koopman

Flow of Presentation

Describe input+measurement noise and its challenges

Solve using polynomial optimization (superstability)

Eliminate noise variables to improve tractability

Extend to other problems (stability models, ARX)

Noise Model and Difficulty

Error-in-Variable Noise Task

Noisy measurements $\mathcal{D} = \{\hat{x}_t, \hat{u}_t\}_{t=1}^{T}$ of linear system

$$x_{t+1} = Ax_t + Bu_t$$

Data $\mathcal{D}$ corrupted by (L_∞-bounded):

Δx : state-measurement noise
Δu : input noise
w : process noise

Find state-feedback $u = Kx$ to stabilize all plants (A, B) consistent with $\mathcal{D}$

Error-in-Variable Relations

Noise processes $\forall t = 1..T$

$$\epsilon_x \;\geq \|\Delta x_t\|_\infty \qquad \epsilon_u \geq \|\Delta u_t\|_\infty \qquad \epsilon_w \geq \|w_t\|_\infty$$

Relations $\forall t = 1..T-1$

$$x_{t+1} = Ax_t + Bu_t + Ew_t$$
$$\hat{x}_t = x_t + \Delta x_t$$
$$\hat{u}_t = u_t + \Delta u_t$$

$(A, B, \Delta x, \Delta u, w)$ unknown, $E \in \mathbb{R}^{n \times e}$ known

Bilinear Trouble

$(A, B, \Delta x, \Delta u, w)$ all unknown

Total of $n(n+m) + T(n+m+e)$ variables

$$\hat{x}_{t+1} - \Delta x_{t+1} = A\hat{x}_t - A\Delta x_t + Bu_t - B\Delta u_t - Ew_t$$

Multiplication between unknown $A\Delta x_t$, also in $B\Delta u_t$

Stabilization task is immediately NP-hard

Even sysid is NP-hard

Consistency Set

Consistency set $\bar{\mathcal{P}}(A, B, \Delta x)$ (with $\epsilon_u = \epsilon_w = 0$)

$$\bar{\mathcal{P}} : \begin{Bmatrix} 0 = -\Delta x_{t+1} + A\Delta x_t + h_t^0 & \forall t = 1..T-1 \\ \|\Delta x_t\|_\infty \leq \epsilon_x & \forall t = 1..T \end{Bmatrix}$$

Affine weight h^0 (residual) is defined by

$$h_t^0 = \hat{x}_{t+1} - A\hat{x}_t - Bu_t \qquad \forall t = 1..T-1$$

Assumption: enough data collected such that $\bar{\mathcal{P}}$ compact

Stability for Plants

Set of plants consistent with $\mathcal{D}$ (with projection π):

$$\mathcal{P}(A, B) = \pi^{A,B}\bar{\mathcal{P}}(A, B, \Delta x)$$

Find $K \in \mathbb{R}^{m \times n}$ such that $(A + BK)$ is Schur $\forall (A, B) \in \mathcal{P}$

Superstability

Superstablity Definition

Superstability (Blanchini and Sznaier 1997, Polyak 2001)

$$\|x\|_\infty \text{ is a CLF : } \qquad \|A + BK\|_\infty < 1$$

Poles of $A + BK$ in unit diamond $\{z \mid \mathrm{Re}(z) + \mathrm{Im}(z) < 1\}$

If $\|A + BK\|_\infty = \gamma$, then $\|x_t\|_\infty \leq \gamma^{(t+1)/n} \|x_0\|_\infty$

Constant K must superstabilize all consistent (A, B)

Superstability Formulations

Linear constraints to impose superstability

Sign-based formulation, $n2^n$ linear constraints

$$\sum_{s\in\{-1,1\}^n} s_j(A+BK)_{ij} < 1 \qquad \forall i$$

Equivalent Convex Lift, $2n^2+n$ linear constraints

$$\begin{aligned} &\exists M \in \mathbb{R}^{n\times n} : \\ &\quad \sum_{j=1}^{m} M_{ij} < 1 && \forall i \\ &\quad -M_{ij} \leq (A+BK)_{ij} \leq M_{ij} && \forall i,j \end{aligned}$$

Process noise only: robust LP (Cheng, Sznaier, Lagoa, 2015)

Full Program

Superstability Application

Superstability certificate $M(A, B) : \mathcal{P} \to \mathbb{R}^{n \times n}$

$2n^2 + n$ inequality expressions over $\mathcal{P}$ (margin $\delta > 0$)

$$\forall i = 1..n : 1 - \delta - \sum_{j=1}^{n} M_{ij}(A, B) \geq 0 \qquad (1a)$$

$$\forall i = 1..n,\ j = 1..n : \qquad (1b)$$

$$M_{ij}(A, B) - (A_{ij} + \sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \geq 0$$

$$M_{ij}(A, B) + (A_{ij} + \sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \geq 0$$

LP in $(M(A, B), K)$ for each $(A, B) \in \mathcal{P}$ (infinite dimensional)

Can choose M to be continuous in compact $\mathcal{P}$

Sum-of-Squares Method

Every $c \in \mathbb{R}$ satisfies $c^2 \geq 0$

Sufficient: $q(x) \in \mathbb{R}[x]$ nonnegative if $q(x) = \sum_i q_i^2(x)$

Exists $v(x) \in \mathbb{R}[x]^s$, *Gram* matrix $Z \in \mathbb{S}_+^s$ with $q = v^T Z v$

Sum-of-Squares (SOS) cone $\Sigma[x]$

$$\begin{aligned} & x^2y^4 - 6x^2y^2 + 10x^2 + 2xy^2 + 4xy - 6x + 4y^2 + 1 \\ =& (x + 2y)^2 + (3x - 1 - xy^2)^2 \end{aligned}$$

Motzkin Counterexample (nonnegative but not SOS)

$$x^2y^4 + x^4y^2 - x^2y^2 + 1$$

Sum-of-Squares Method (cont.)

Putinar Positivestellensatz (Psatz) nonnegativity certificate over set $\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}$:

$$q(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x)$$
$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma_i(x) \in \Sigma[x], \quad \phi_j \in \mathbb{R}[x]$$

Psatz at degree $2d$ is an SDP, monomial basis: $s = \binom{n+d}{d}$

Archimedean: $\exists R \geq 0$ where $R - \|x\|_2^2$ has Psatz over $\mathbb{K}$

Computational Complexity (Full)

Restrict $M_{ij}(A, B)$ to a polynomial of degree $2d$

Each infinite-dimensional linear constraint becomes an SOS constraint (Psatz) in $(A, B, \Delta x)$: $\Sigma[\bar{\mathcal{P}}]$

Each Psatz has a PSD Gram matrix of size $\binom{n(n+m+T)+d}{d}$

$(n = 2, m = 2, T = 15, d = 2)$: size 780

Alternatives

Motivation and Size Comparison

Use Δx-affine structure of $\bar{\mathcal{P}}$ to eliminate Δx

Maximal size of Gram (PSD) matrices

Size	Full	Alternatives
Super	$\binom{n(n+m+T)+d}{d}$	$\binom{n(n+m)+d}{d}$

When $(n = 2, m = 2, T = 15, d = 2)$:

$$\text{Full} = 780, \ \text{Altern.} = 45$$

Robust Counterpart Method (eliminating noise)

Linear inequality involving Δx

$$q(A,B) \geq 0 \qquad \forall (A,B,\Delta x) \in \bar{\mathcal{P}}$$

Polytope-constrained noise Δx

$$\Delta x \in \bar{\mathcal{P}} = \{\Delta x \mid G\Delta x \leq h,\ C\Delta x = f\}$$

All (q, G, h, C, f) are functions of (A, B) $\bar{\mathcal{P}}$

Robust Counterpart without Δx (equivalent)

$$\exists \zeta \geq 0, \mu \mid q \geq h^T\zeta + f^T\mu,\ 0 = G^T\zeta + C^T\mu.$$

Theorem of Alternatives

Superstability condition q: Full program in $(A, B, \Delta x)$

$$q(A,B) \geq 0 \qquad \forall (A, B, \Delta x) \in \bar{\mathcal{P}}$$

Alternatives program in (A, B) with no conservatism

$$\begin{aligned}
\text{find} \quad & \zeta^{\pm}_{1:T}(A,B) \geq 0, \ \mu_{1:T-1}(A,B) \\
& q \geq \textstyle\sum_{t,i} \epsilon_x (\zeta^{+}_{t,i} + \zeta^{-}_{t,i}) + \sum_{t=1}^{T-1} \mu_t^T h_t^0 \quad \forall (A,B) \\
& \zeta^{+}_{1} - \zeta^{-}_{1} = A^T \mu_1 \\
& \zeta^{+}_{T} - \zeta^{-}_{T} = -\mu_{T-1} \\
& \zeta^{+}_{t} - \zeta^{-}_{t} = A^T \mu_t - \mu_{t-1} \qquad \forall t \in 2..T-1
\end{aligned}$$

Polynomial Alternatives Certificate

Choose $\zeta^{\pm}$ SOS, μ polynomial when $\bar{\mathcal{P}}$ compact

Express SOS Alternatives certificate as $q(A, B) \in \Sigma^{\text{alt}}[\mathcal{P}]$

Find degree-$2d$ polynomial matrix $M_{ij}(A, B)$ with

$$\begin{aligned}
&\forall i = 1..n : 1 - \delta - \textstyle\sum_{j=1}^{n} M_{ij}(A, B) \in \Sigma^{\text{alt}}[\mathcal{P}] \\
&\forall i = 1..n, \ j = 1..n : \\
&\quad M_{ij}(A, B) - (A_{ij} + \textstyle\sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \in \Sigma^{\text{alt}}[\mathcal{P}] \\
&\quad M_{ij}(A, B) + (A_{ij} + \textstyle\sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \in \Sigma^{\text{alt}}[\mathcal{P}]
\end{aligned}$$

$\zeta^{\pm}, \ \mu$: same multiplicity as SOS Psatz multipliers over $\bar{\mathcal{P}}$

Further notes about complexity

In practice $d = 1$ suffices for Alternatives while $d = 2$ is required for Full

With $(n = 2, m = 1, d_{\mathrm{alt}} = 1, d_{\mathrm{full}} = 2)$

Maximum size PSD matrices

	Gram	ζ	μ (vector)
Alternatives	7	7	7
Full (T = 4)	120	15	120
Full (T = 6)	190	19	190
Full (T = 8)	276	23	276

All Noise

All Noise Consistency Set

Consistency set $\bar{\mathcal{P}}^{\text{all}}(A, B, \Delta x, \Delta u, w)$:

$$
\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + Ew_t && \forall t = 1..T-1 \\
\hat{x}_t &= x_t + \Delta x_t, \quad \hat{u}_t = u_t + \Delta u_t && \forall t = 1..T-1 \\
\epsilon_x &\geq \|\Delta x_t\|_\infty,\ \epsilon_u \geq \|\Delta u_t\|_\infty,\ \epsilon_w \geq \|w_t\|_\infty && \forall t = 1..T
\end{aligned}
$$

Set of consistent plants,

$$
\mathcal{P}^{\text{all}}(A, B) = \pi^{A,B}\bar{\mathcal{P}}^{\text{all}}(A, B, \Delta x, \Delta u, w)
$$

$(\Delta x, \Delta u, w)$ together not much more complex than Δx alone

All Noise Size

Use Alternatives to eliminate $(\Delta x, \Delta u, w)$

Maximal size of Gram (PSD) matrices

Size	Full	Alternatives
Super	$\binom{n(n+m)+T(n+m+e)+d}{d}$	$\binom{n(n+m)+d}{d}$

When $(n = 2, m = 2, T = 15, d = 2, e = 1)$:

Full $= 3570$, Alternatives $= 45$

Other Stabilization Methods

Extended Superstability

Weights $v > 0$ matrix $Y = \text{diag}(v)$ (Polyak 2004)

CLF $\|x./v\|_\infty$ if $\|Y(A + BK)Y^{-1}\|_\infty < 1$

Find $v \in \mathbb{R}^n_{>0},\ S \in \mathbb{R}^{m \times n}, M : \mathcal{P} \to \mathbb{R}^{n \times n}$ with $\forall (A, B) \in \mathcal{P}$:

$$\begin{aligned} &\textstyle\sum_{j=1}^n M_{ij} < v_i && \forall i \in 1..n \\ &- M_{ij} \leq A_{ij} v_j + \textstyle\sum_{k=1} B_{ik} S_{kj} \leq M_{ij} && \forall i, j \in 1..n \end{aligned}$$

Return $K = SY^{-1}$, has $2n^2 + n$ Psatz constraints

Positive Stabilization

Positive System: keeps $\mathbb{R}^n_{\geq 0}$ invariant under $u \in \mathbb{R}^m_{\geq 0}$

Weights $v > 0$ matrix $Y = \text{diag}(v)$ (Ait Rami 2008)

Dual Linear Copositive Lyapunov Function $\max_i (x_i / v_i)$

Find $v \in \mathbb{R}^n_{>0}, \; S \in \mathbb{R}^{m \times n}$ with $\forall (A, B) \in \mathcal{P}$:

$$\begin{aligned} & v - (AY + BS)\mathbf{1} \in \mathbb{R}^n_{>0} \\ & AY + BS \in \mathbb{R}^{n \times n}_{\geq 0} \end{aligned}$$

Return $K = SY^{-1}$, has $n^2 + n$ Psatz constraints

Quadratic Stabilization

Quadratic Lyapunov function $x^T Yx$ for $Y \in \mathbb{S}^n_{++}$

$$Q(A, B) = \begin{bmatrix} Y & (A + BK)Y \\ * & Y \end{bmatrix} = \begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \in \mathbb{S}^{2n}_{++}$$

Recover controller $K = SY^{-1}$

Find constant (Y, K) to stabilize all $(A, B) \in \mathcal{P}$

Polynomial Matrix Inequalities

SOS method (scalar): $q(x) \geq 0$

Extend to matrices $Q(x) \in \mathbb{S}^s_{++}$

SOS matrix: $Q(x) = R(x)^T R(x) \in \Sigma^s[x]$ for matrix $R(x)$

Gram matrix (PSD) constraint of size $s\binom{n+d}{d}$

Scherer Psatz: nonnegativity over constraint sets

Quadratic Stabilization Program

Quadratic Full: Size $2n\binom{n(n+m+T)+d}{d}$

$$\begin{bmatrix} Y & AY + BS \\ * & Y \end{bmatrix} \in \Sigma^{2n}[\bar{\mathcal{P}}]_{\leq 2d} \tag{2}$$

Can eliminate Δx, form Alternatives with size $2n\binom{n(n+m)+d}{d}$

Alternatives could add conservatism

Extend to worst-case-H_2-optimal control

Single-Input Single-Output

ARX model

Autoregressive Model with Exogenous Input (ARX)

$$y_t = -\sum_{i=1}^{n_a} a_i y_{t-i} + \sum_{i=1}^{n_b} b_i u_{t-i}$$

Data $\mathcal{D} = (\hat{u}, \hat{y})$ and no state x,

$$\begin{aligned} \hat{u} &= u + \Delta u, & \|\Delta u\|_\infty &\leq \epsilon_u \\ \hat{y} &= y + \Delta y, & \|\Delta y\|_\infty &\leq \epsilon_y \end{aligned}$$

Find controller u to stabilize (a, b) consistent with $\mathcal{D}$

Superstability for ARX

Original model with vectors (a, b)

$$y_t = -\sum_{i=1}^{n_a} a_i y_{t-i} + \sum_{i=1}^{n_b} b_i u_{t-i}.$$

Transfer Function with one-step-behind operator $\lambda u_t = u_{t-1}$

$$G(\lambda) = \frac{\sum_{i=1}^{n_b} b_i \lambda^i}{1 + \sum_{i=1}^{n_a} a_i \lambda^i} = \frac{B(\lambda)}{1 + A(\lambda)}$$

Superstability definition, linear constraints

$$\|a\|_1 < 1$$

Dynamic Compensation

Compensator $C(\lambda) = \tilde{B}(\lambda)/(1 + \tilde{A}(\lambda))$

Closed-loop system

$$G_{cl}(\lambda) = \frac{G(\lambda)}{1 + G(\lambda)C(\lambda)} = \frac{B(\lambda)(1 + \tilde{A}(\lambda))}{(1 + A(\lambda))(1 + \tilde{A}(\lambda)) + B(\lambda)\tilde{B}(\lambda)}$$

Superstable: coefficients of G_{cl} denominator have L_1 norm < 1

Fixed C superstabilizes all $(A, B) \in \mathcal{P}$ (from $\mathcal{D}$)

ARX Program Sizes

Set $\mathcal{P}$ originally contains $(a, b, \Delta u, \Delta y)$

Eliminate $(\Delta u, \Delta y)$ in alternatives

Maximal size of Gram (PSD) matrices ($N = N_a + N_b$)

Size	Full	Alternatives
Super	$\binom{2N+T-1+d}{d}$	$\binom{N+d}{d}$

No conservatism in Alternatives

Examples

Example 1

Ground-truth system $n = 3, m = 2, T = 40$

$$A = \begin{bmatrix} 0.6852 & 0.0274 & 0.5587 \\ 0.2045 & 0.6705 & 0.1404 \\ 0.8781 & 0.4173 & 0.1981 \end{bmatrix}, \ B = \begin{bmatrix} 0.4170 & 0.3023 \\ 0.7203 & 0.1468 \\ 0.0001 & 0.0923 \end{bmatrix}$$

Noise parameters $\epsilon_x = 0.05, \epsilon_u = 0, \ \epsilon_w = 0$

Solve $\gamma^* = \min_{\gamma \in \mathbb{R}} \gamma : \|A + BK\|_\infty \leq \gamma$ for all $(A, B) \in \mathcal{P}$

Example 1: Complexity

Data horizon $T = 6$,

	d	#scalar variables
Full	2	3.4×10^7
Altern.	1	67776

Altern recovers ground truth $\gamma^* = 0.7259$ when $\epsilon_x = 0$

Example 1: Results

With $T = 40$:

$\gamma^*_{\text{alt}} = 0.8880$ Alternatives with $d = 1$ (worst-case)

$\gamma^*_{\text{clp}} = 0.7749$ Alternatives controller applied to ground truth

$\gamma^*_{\text{true}} = 0.7259$ Ground truth

Example 2: (Monte Carlo, Superstabilization)

Ground truth system $(\epsilon_w, \epsilon_u = 0)$

$$A = \begin{bmatrix} 0.6863 & 0.3968 \\ 0.3456 & 1.0388 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4170 & 0.0001 \\ 0.7203 & 0.3023 \end{bmatrix}$$

S = percentage of success in 50 trials

S vs. ϵ_x with $T = 8$

ϵ_x	0.05	0.08	0.11	0.14
S	100	84	57	39

S vs. T with $\epsilon_x = 0.14$

T	8	10	12	14
S	39	60	75	86

Example 3: (Monte Carlo, Stabilization)

(Extended) Super, Positive, and Quadratic Stability

Success vs. ϵ_x with $T = 8$

ϵ	0.05	0.08	0.11	0.14
ESS	100	88	69	40
SS	100	84	57	39
PS	94	61	19	3
QS	100	100	90	79

Success vs. T with $\epsilon_x = 0.14$

T	8	10	12	14
ESS	40	61	78	89
SS	39	60	75	86
PS	3	20	42	56
QS	79	86	95	99

Example 4: (Monte Carlo, H2 Performance)

Median H_2 performance in 100 trials (PMI)

H_2-norm vs. ϵ_x with $T = 8$

ϵ	0.05	0.08	0.11	0.14
$\gamma_{2,\mathrm{clp}}$	1.97	2.07	2.18	2.15
$\gamma_{2,\mathrm{worst}}$	2.30	2.73	3.23	4.31

H_2-norm vs. T with $\epsilon_x = 0.14$

T	8	10	12	14
$\gamma_{2,\mathrm{clp}}$	2.07	1.96	1.94	1.93
$\gamma_{2,\mathrm{worst}}$	2.73	2.42	2.23	2.20

Example 5: (ARX Superstabilization)

Ground truth system ($\epsilon_w, \epsilon_u = 0$)

$$y_t = u_{t-2} - (0.5y_{t-1} - 1.21y_{t-2} - 0.605y_{t-3})$$

Fixed-order control $n_a = 4, n_b = 3$ with $\epsilon_y = \epsilon_u = \epsilon$

γ v.s. ϵ with $T = 80$

ϵ	0.02	0.04	0.06	0.08
γ	0.25	0.49	0.73	0.98

γ v.s. T with $\epsilon = 0.02$

T	20	40	60	80
γ	0.44	0.31	0.27	0.25

Take-aways

Conclusion

Stabilization in the Error-in-variables setting

Formulate SOS certificates over consistency set

Alternatives to simplify computational complexity

Conservatism only introduced in Quadratic Stability

Thank you for your attention

Bonus Content

Set Membership: Process Noise Alone

Superstability with only L_∞-bounded process noise (not EIV)

$$\hat{x}_{k+1} = A\hat{x}_t + B\hat{u}_t + w_t \qquad\qquad \forall t = 1..T-1$$

Polytope of data-consistent plants $P_1(A, B)$:

$$P_1 = (A, B): \ \|\hat{x}_{k+1} - A\hat{x}_t - B\hat{u}_t\|_\infty \leq \epsilon_w \quad \forall t = 1..T-1$$

Superstable-plants polytope $P_2(A, B)$ given constant (M, K)

$$P_2 = (A, B): \ -M \leq A + BK \leq M$$

Control via LP (Cheng, Sznaier, Lagoa 2015)

Sparse but Conservative Tightening

Equality constraints $0 = -\Delta x_{t+1} + A\Delta x_t + h_t^0$

Define row groups $C_i = (A_{i,1:n}, B_{i,1:m})$

Each equality constraint in (i, t) only involves one group

Sparse multipliers $\zeta_{it}^{\pm}(C_i) \geq 0,\ \mu_{it}(C_i)$

Max. Gram matrix size $\binom{n+m+d}{d}$ rather than $\binom{n(n+m)+d}{d}$

Has never worked on our experiments though