Data-Driven Superstabilizing Control of Error-In-Variables Discrete-Time Linear Systems

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Error-in-Variable Noise Task

Noisy measurements $\mathcal{D} = \{\hat{x}_t, \hat{u}_t\}_{t=1}^{T}$ of linear system

$$x_{t+1} = Ax_t + Bu_t$$

Data $\mathcal{D}$ corrupted by (L_∞-bounded):

Δx : state-measurement noise

Δu : input noise

w : process noise

Find state-feedback $u = Kx$ to stabilize all plants (A, B) consistent with $\mathcal{D}$

Error-in-Variable Relations

Noise processes $\forall t = 1..T$

$$\epsilon_x \geq \|\Delta x_t\|_\infty \qquad \epsilon_u \geq \|\Delta u_t\|_\infty \qquad \epsilon_w \geq \|w_t\|_\infty$$

Relations $\forall t = 1..T-1$

$$x_{t+1} = Ax_t + Bu_t + Ew_t$$
$$\hat{x}_t = x_t + \Delta x_t$$
$$\hat{u}_t = u_t + \Delta u_t$$

$(A, B, \Delta x, \Delta u, w)$ unknown, $E \in \mathbb{R}^{n \times e}$ known

Bilinear Trouble

$(A, B, \Delta x, \Delta u, w)$ all unknown

Total of $n(n+m) + T(n+m+e)$ variables

$$\hat{x}_{t+1} - \Delta x_{t+1} = A\hat{x}_t - A\Delta x_t + Bu_t - B\Delta u_t - Ew_t$$

Multiplication between unknown $A\Delta x_t$, also in $B\Delta u_t$

Stabilization task is immediately NP-hard

Even sysid is NP-hard

Main Ideas

Use superstability to form a more tractable control problem

Formulate a large-scale polynomial optimization problem

Improve scalability by applying a Theorem of Alternatives

Superstability

Superstablity Definition

Superstability (Polyak 2001), $\|x\|_\infty$ is a CLF

$$\|A + BK\|_\infty < 1$$

Poles of $A + BK$ in unit diamond $\{z \mid \mathrm{Re}(z) + \mathrm{Im}(z) < 1\}$

If $\|A + BK\|_\infty = \gamma$, then $\|x_t\|_\infty \leq \gamma^{(t+1)/n} \|x_0\|_\infty$

Constant K must superstabilize all consistent (A, B)

Superstability Formulations

Linear constraints to impose superstability

Sign-based formulation, $n2^n$ linear constraints

$$\sum_{s\in\{-1,1\}^n} s_j(A+BK)_{ij} < 1 \qquad \forall i$$

Equivalent Convex Lift, $2n^2+n$ linear constraints

$$\begin{aligned}
&\exists M \in \mathbb{R}^{n\times n}: \\
&\quad \sum_{j=1}^{m} M_{ij} < 1 && \forall i \\
&\quad -M_{ij} \le (A+BK)_{ij} \le M_{ij} && \forall i,j
\end{aligned}$$

Full Program

Consistency Set

Consistency set $\bar{\mathcal{P}}(A, B, \Delta x)$ (with $\epsilon_u = \epsilon_w = 0$)

$$\bar{\mathcal{P}}: \begin{Bmatrix} 0 = -\Delta x_{t+1} + A\Delta x_t + h_t^0 & \forall t = 1..T-1 \\ \|\Delta x_t\|_\infty \leq \epsilon_x & \forall t = 1..T \end{Bmatrix}$$

Affine weight h^0 is defined by,

$$h_t^0 = \hat{x}_{t+1} - A\hat{x}_t - Bu_t \qquad \forall t = 1..T-1.$$

Assumption: enough data collected such that $\bar{\mathcal{P}}$ compact

Superstability for Plants

Set of plants consistent with $\mathcal{D}$ (with projection π):

$$\mathcal{P}(A, B) = \pi^{A,B}\bar{\mathcal{P}}(A, B, \Delta x)$$

Find $K \in \mathbb{R}^{m \times n}$ such that $(A + BK)$ is Schur $\forall (A, B) \in \mathcal{P}$

Restrict to superstability: $\|A + BK\|_\infty < 1, \quad \forall (A, B) \in \mathcal{P}$

Superstability Application

Superstability certificate $M(A, B) : \mathcal{P} \to \mathbb{R}^{n \times n}$

$2n^2 + n$ inequality expressions over $\bar{\mathcal{P}}$ (margin $\delta > 0$)

$$\forall i = 1..n : 1 - \delta - \textstyle\sum_{j=1}^{n} M_{ij}(A, B) \geq 0 \tag{1a}$$

$$\begin{aligned} &\forall i = 1..n,\ j = 1..n : \\ &\quad M_{ij}(A, B) - (A_{ij} + \textstyle\sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \geq 0 \\ &\quad M_{ij}(A, B) + (A_{ij} + \textstyle\sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \geq 0 \end{aligned} \tag{1b}$$

Can choose M to be continuous in compact $\mathcal{P}$

Computational Complexity (Full)

Restrict $M_{ij}(A, B)$ to a polynomial of degree $2d$

Each infinite-dimensional linear constraint becomes an SOS constraint (Psatz) in $(A, B, \Delta x)$

Each Psatz has a PSD Gram matrix of size $\binom{n(n+m+T)+d}{d}$

$(n = 2, m = 2, T = 15, d = 2)$: size 780

Alternatives

Motivation and Size Comparision

Use Δx-affine structure of $\bar{\mathcal{P}}$ to eliminate Δx

Maximal size of Gram (PSD) matrices

Size	Full	Alternatives
Super	$\binom{n(n+m+T)+d}{d}$	$\binom{n(n+m)+d}{d}$

When $(n = 2, m = 2, T = 15, d = 2)$:

$$\text{Full} = 780, \text{ Altern.} = 45$$

Theorem of Alternatives

Superstability condition q: Full program in $(A, B, \Delta x)$

$$q(A, B) \geq 0 \qquad \forall (A, B, \Delta x) \in \bar{\mathcal{P}}$$

Alternatives program in (A, B) with no conservatism

$$\begin{aligned}
\text{find} \quad & \zeta^{\pm}_{1:T}(A, B) \geq 0,\ \mu_{1:T-1}(A, B) \\
& q \geq \textstyle\sum_{t,i} \epsilon_x(\zeta^{+}_{t,i} + \zeta^{-}_{t,i}) + \sum_{t=1}^{T-1} \mu_t^T h_t^0 \quad \forall (A, B) \\
& \zeta^{+}_1 - \zeta^{-}_1 = A^T \mu_1 \\
& \zeta^{+}_T - \zeta^{-}_T = -\mu_{T-1} \\
& \zeta^{+}_t - \zeta^{-}_t = A^T \mu_t - \mu_{t-1} \qquad \forall t \in 2..T-1
\end{aligned}$$

Polynomial Alternatives Certificate

Choose $\zeta^{\pm}$ SOS, μ polynomial when $\bar{\mathcal{P}}$ compact

Express SOS Alternatives certificate as $q(A, B) \in \Sigma^{\text{alt}}[\mathcal{P}]$

Find degree-$2d$ polynomial matrix $M_{ij}(A, B)$ with

$$\forall i = 1..n : 1 - \delta - \sum_{j=1}^{n} M_{ij}(A, B) \in \Sigma^{\text{alt}}[\mathcal{P}]$$

$$\forall i = 1..n,\ j = 1..n :$$

$$M_{ij}(A, B) - (A_{ij} + \sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \in \Sigma^{\text{alt}}[\mathcal{P}]$$

$$M_{ij}(A, B) + (A_{ij} + \sum_{\ell=1}^{m} B_{i\ell} K_{\ell j}) \in \Sigma^{\text{alt}}[\mathcal{P}]$$

$\zeta^{\pm},\ \mu$: same multiplicity as SOS Psatz multipliers over $\bar{\mathcal{P}}$

Further notes about complexity

In practice $d = 1$ suffices for Alternatives while $d = 2$ is required for Full

With $(n = 2, m = 1, d_{\mathrm{alt}} = 1, d_{\mathrm{full}} = 2)$

Maximum size PSD matrices

	Gram	ζ	μ (vector)
Alternatives	7	7	7
Full (T = 4)	120	15	120
Full (T = 6)	190	19	190
Full (T = 8)	276	23	276

All Noise

All Noise Consistency Set

Consistency set $\bar{\mathcal{P}}^{\text{all}}(A, B, \Delta x, \Delta u, w)$:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + Ew_t && \forall t = 1..T-1 \\ \hat{x}_t &= x_t + \Delta x_t, \quad \hat{u}_t = u_t + \Delta u_t && \forall t = 1..T-1 \\ \epsilon_x &\geq \|\Delta x_t\|_\infty, \; \epsilon_u \geq \|\Delta u_t\|_\infty, \; \epsilon_w \geq \|w_t\|_\infty && \forall t = 1..T \end{aligned}$$

Set of consistent plants,

$$\mathcal{P}^{\text{all}}(A, B) = \pi^{A,B} \bar{\mathcal{P}}^{\text{all}}(A, B, \Delta x, \Delta u, w)$$

$(\Delta x, \Delta u, w)$ together not much more complex than Δx alone

All Noise Size

Use Alternatives to eliminate $(\Delta x, \Delta u, w)$

Maximal size of Gram (PSD) matrices

Size	Full	Alternatives
Super	$\binom{n(n+m)+T(n+m+e)+d}{d}$	$\binom{n(n+m)+d}{d}$

When $(n = 2, m = 2, T = 15, d = 2, e = 1)$:

Full $= 3570$, Alternatives $= 45$

Examples

Example 1

Ground-truth system $n = 3, m = 2, T = 40$

$$A = \begin{bmatrix} 0.6852 & 0.0274 & 0.5587 \\ 0.2045 & 0.6705 & 0.1404 \\ 0.8781 & 0.4173 & 0.1981 \end{bmatrix}, \ B = \begin{bmatrix} 0.4170 & 0.3023 \\ 0.7203 & 0.1468 \\ 0.0001 & 0.0923 \end{bmatrix}$$

Noise parameters $\epsilon_x = 0.05, \epsilon_u = 0, \ \epsilon_w = 0$

Solve $\gamma^* = \min_{\gamma \in \mathbb{R}} \gamma : \|A + BK\|_\infty \leq \gamma$ for all $(A, B) \in \mathcal{P}$

Example 1: Complexity

Data horizon $T = 6$,

	d	#scalar variables
Full	2	3.4×10^7
Altern.	1	67776

Altern recovers ground truth $\gamma^* = 0.7259$ when $\epsilon_x = 0$

Example 1: Results

With $T = 40$:

$$\begin{aligned} \gamma^*_{\text{alt}} &= 0.8880 \quad \text{Alternatives with } d = 1 \text{ (worst-case)} \\ \gamma^*_{\text{clp}} &= 0.7749 \quad \text{Alternatives controller applied to ground truth} \\ \gamma^*_{\text{true}} &= 0.7259 \quad \text{Ground truth} \end{aligned}$$

Example 2: (Monte Carlo test)

Ground truth system $(\epsilon_w, \epsilon_u = 0)$

$$A = \begin{bmatrix} 0.6863 & 0.3968 \\ 0.3456 & 1.0388 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4170 & 0.0001 \\ 0.7203 & 0.3023 \end{bmatrix}$$

S = number of successful designs out of 100 trials

S vs. ϵ_x with $T = 8$

ϵ_x	0.05	0.08	0.11	0.14
S	100	84	57	39

S vs. T with $\epsilon_x = 0.14$

T	8	10	12	14
S	39	60	75	86

Take-aways

Conclusion

Superstabilization in the Error-in-variables setting

Formulate SOS certificates over consistency set

Alternatives to simplify computational complexity

Conservatism only introduced in Superstability

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github:jarmill/error_in_variables

Bonus Content

Sum-of-Squares Method

Nonnegative $q(x) \in \mathbb{R}[x]$ is SOS ($q \in \Sigma[x]$) if there exists a vector $v(x) \in \mathbb{R}[x]^s$, *Gram* matrix $Z \in \mathbb{S}_+^s$ with $q = v^T Z v$

Putinar Positivestellensatz (Psatz) nonnegativity certificate over set $\mathbb{K} = \{x \mid g_i(x) \geq 0, h_j(x) = 0\}$:

$$q(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x) + \sum_j \phi_j(x) h_j(x)$$
$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma_i(x) \in \Sigma[x], \quad \phi_j \in \mathbb{R}[x].$$

Psatz at degree $2d$ is an SDP, monomial basis: $s = \binom{n+d}{d}$