Bounding the Distance of Closest Approach to Unsafe Sets with Occupation Measures

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***Abstract*— This paper presents a method to lower-bound the distance of closest approach between points on an unsafe set and points along system trajectories. Such a minimal distance is a quantifiable and interpretable certificate of safety of trajectories, as compared to prior art in barrier and density methods which offers a binary indication of safety/unsafety. The distance estimation problem is converted into a infinite-dimensional linear program in occupation measures based on existing work in peak estimation and optimal transport. The moment-SOS hierarchy is used to obtain a sequence of lower bounds obtained through solving semidefinite programs in increasing size, and these lower bounds will converge to the true minimal distance as the degree approaches infinity under mild conditions (e.g. Lipschitz dynamics, compact sets).**

I. Introduction

A trajectory $x(t \mid x_0)$ lying in the space $X \subseteq \mathbb{R}^n$ of the dynamical system $\dot{x}(t) = f(t, x(t))$ starting from an initial condition $x_0 \in X_0$ is safe with respect to the closed unsafe set X_u if $x(t \mid x_0) \notin X_u$ for all times t between $t = 0$ and the time horizon $t = T$. The safety of trajectories starting from X_0 may be quantified by the distance of closest approach given a distance function $c(x, y)$ as,

$$\begin{aligned} P^* = \min_{t,\, x_0, y} \ & c(x(t \mid x_0), y) \\ & \dot{x}(t') = f(t', x), \quad \forall t' \in [0, T] \\ & t \in [0, T], \ x_0 \in X_0, \ y \in X_u. \end{aligned} \tag{1}$$

The task of *distance estimation* will refer to solving Problem (1), and *distance bounding* will mean to find a lower bound $p^* \leq P^*$ that is as tight as possible. A direct solution to (1) in terms of optimizing over (t, x_0, y) is generically difficult and non-convex. This paper will propose a reformulation of (1) into a convex infinite-dimensional Linear Program (LP) of nonnegative Borel measures based on the occupation measure work in optimal control [1], [2], optimal transport methods [3], [4], and peak estimation [5], [6]. The infinite-dimensional LP will be truncated into a sequence of finite-dimensional Semidefinite Programs (SDPs) in increasing complexity through the moment-Sum of Squares (SOS) hierarchy [7]. Prior work on verifying safety of trajectories include barrier functions [8], [9] density functions [10], forward-backward reachability [11], and interval analysis [12], [13], but these methods do not yield a measure of proximity to the unsafe set. The work in [14] introduced the concept of safety margins as a measurement of constraint violation (computed through maximin peak estimation), but safety margins are difficult to interpret and will scale as the parameterization of the constraint set changes (even in the same coordinate system). The distance of closest approach is an intuitive geometric quantification of the safety of trajectories.

Contributions of this paper include:

- An LP in measures to lower bound (1)
- A proof that bounds obtained from the moment-SOS hierarchy will converge to P^* as the degree approaches infinity under mild conditions
- An extension to performing distance estimation for systems with dynamic uncertainty

This paper has the following structure: Section II will cover preliminaries such as notation, the moment-SOS hierarchy, and occupation measure methods for peak estimation and safety analysis. Section III will present and discuss an infinite-dimensional LP in occupation measures to perform the distance estimation task along with its Linear Matrix Inequality (LMI) truncation. Section IV will demonstrate effectiveness of this LMI method on examples of distance estimation. Section V will briefly highlight extensions to the distance estimation framework. Section VI will conclude the paper. An extended version of this paper (including detailed proofs, correlative sparsity, and certifying distance of shapes) is available at https://arxiv.org/abs/2110.14047 [15].

II. Preliminaries

A. Acronyms/Initialisms

LMI	Linear Matrix Inequality
LP	Linear Program
ODE	Ordinary Differential Equation
PSD	Positive Semidefinite
SDP	Semidefinite Program
SOS	Sum of Squares

B. Notation and Measure Theory

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{N}^n$ be the set of natural number multi-indices in n terms. The point-set distance $c(x, Y)$ given a metric c and a set Y is defined as $\min_{y \in Y} c(x, y)$. The set $\mathbb{N}^n_{\leq d}$ for fixed positive integral d is the finite set of multi-indices α where $\sum_{i=1}^n \alpha_i \leq d$. The set of polynomials with real coefficients in indeterminates x is

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$\mathbb{R}[x]$, and a polynomial $p(x)$ may be represented as $p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \prod_{i=1}^n x_i^{\alpha_i}$ where a finite number of the constant coefficients p_α are nonzero. The degree of a monomial is $\deg(x^\alpha) = |\alpha| = \sum_{i=1}^n \alpha$, and the degree of a polynomial is $\max_{\alpha \in \mathbb{N}^n} |\alpha|$ such that $p_\alpha \neq 0$. A matrix Q is Positive Semidefinite (PSD) if its quadratic form satisfies $x^T Q x \succeq 0, \forall x \neq 0$, and this will be expressed by $Q \succeq 0$.

The set of continuous functions on a space X is $C(X)$, and its subset of functions with continuous k-th derivatives is $C^k(X)$. The subcone of nonnegative continuous functions $C_+(X)$ is dual to the set of nonnegative Borel measures $\mathcal{M}_+(X)$ supported on X. An inner product $\langle f_+, \mu \rangle = \int_X f_+(x) d\mu(x)$ is defined by Lebesgue integration for $f_+ \in C_+(X),\ \mu \in \mathcal{M}_+(X)$, and this inner product is generalized to a duality pairing $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$ between $f \in C(X),\ \mu \in \mathcal{M}_+(X)$. The mass of a measure is $\langle 1, \mu \rangle$, and a probability measure has mass 1. The indicator function $I_A(x)$ of a set $A \subseteq X$ has value 0 when $x \notin A$ and value 1 when $x \in A$, and the measure of A is $\mu(A) = \langle I_A(x), \mu \rangle$. The support of μ is the set of $x' \in X$ where every open neighborhood $N(x')$ has $\mu(N(x')) > 0$. A rank-r atomic measure is supported at r distinct points, and these support points are called atoms. The Dirac delta $\delta_{x'}$ with pairing $\langle f, \delta_{x'} \rangle = f(x')$ is a rank-1 atomic probability measure supported only at x'.

The unique product measure $\mu \otimes \nu \in \mathcal{M}_+(X \times Y)$ given $\mu \in \mathcal{M}_+(X),\ \nu \in \mathcal{M}_+(Y)$ satisfies $\forall A \in X,\ B \in Y : (\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$. The projection map $\pi^x : X \times Y \to X$ performs $\pi((x, y)) = x$. The marginalization operator $\pi^x_\#$ yields the x-marginal of a measure $\eta \in \mathcal{M}_+(X \times Y)$ as $\langle w(x), \eta(x, y) \rangle = \langle w(x), \pi^x_\# \eta(x) \rangle,\ \forall w(x) \in C(X)$. All linear operators $\mathcal{L} : X \to Y$ have unique adjoint operators $\mathcal{L}^\dagger : Y^* \to X^*$ such that $\langle \mathcal{L} f, \mu \rangle = \langle f, \mathcal{L}^\dagger \mu \rangle,\ \forall f \in C(X),\ \mu \in \mathcal{M}_+(X)$.

C. Moment-SOS Hierarchy

Refer to [7] for more detail about all aspects reviewed in this subsection. An LP in a measure $\mu \in \mathcal{M}_+(X)$ is a convex optimization in terms of a cost $p(x) \in C(X)$, a set of constraint functions $a_j(x) \in C(X)$, and answer values b_j for $j = 1, \ldots, J_{max}$ of the form:

$$p^* = \sup_{\mu \in \mathcal{M}_+(X)} \langle p, \mu \rangle \tag{2a}$$

$$\langle a_j(x), \mu \rangle = b_j \qquad \forall j = 1, \ldots, J_{max}. \tag{2b}$$

The α-moment of $\mu \in \mathcal{M}_+(X)$ is the inner product $m_\alpha = \langle x^\alpha, \mu \rangle$ for a multi-index $\alpha \in \mathbb{N}^n$. A measure μ is bounded if all of its moments m_α are bounded for $|\alpha| < \infty$. Sufficient conditions for μ to be bounded are that $\langle 1, \mu \rangle$ is finite and the set X is compact.

Assume for the remainder of this section that (p, a_j) are polynomials and that the set X is a basic semialgebraic set $X = \{x \in \mathbb{R}^n \mid g_k(x) \geq 0\ \forall k = 1, \ldots, N_c\}$, which is a set formed by the intersection of a finite number of polynomial inequality constraints where the degree of each $g_k(x)$ is bounded. Let $\mathbf{m} = \{\mathbf{m}_\alpha\}_{\alpha \in \mathbb{N}}$ be an infinite moment sequence. Define $\mathbb{M}[X\mathbf{m}] = \mathrm{diag}(\mathbb{M}[\mathbf{m}], \{\mathbb{M}[g_k \mathbf{m}]\}_{k=1}^{N_c})$ as a block-diagonal matrix comprising the moment matrix $\mathbb{M}[\mathbf{m}]$ and localizing matrices $\mathbb{M}[g_k \mathbf{m}]$,

$$\mathbb{M}[\mathbf{m}]_{\alpha,\beta} = m_{\alpha+\beta} \quad \mathbb{M}[g_k \mathbf{m}]_{\alpha,\beta} = \textstyle\sum_{\gamma \in \mathbb{N}^n} g_{k\gamma} m_{\alpha+\beta+\gamma}.$$

There exists some measure $\mu \in \mathcal{M}_+(X)$ (called a representing measure associated with $\mathbf{m}$) that agrees with moment sequence as $\mathbf{m}_\alpha = \langle x^\alpha, \mu \rangle$ if the set X satisfies an Archimedean condition and when $\mathbb{M}_d[X\mathbf{m}]$ is PSD [16]. Appending a redundant ball constraint $R^2 - \|x\|^2 \geq 0$ for sufficiently large R to the inequality description of a compact X will ensure that X satisfies this Archimedean property. The degree-d truncation of a moment matrix $\mathbb{M}_d[m]$ for a positive integer d is a finite dimensional matrix with size $\binom{n+d}{d}$ including moments only up to order $2d$. The degree-d LMI relaxation of the LP (2) is,

$$p_d^* = \max_{\mathbf{m}} \textstyle\sum_\alpha p_\alpha y_\alpha \tag{3a}$$

$$\mathbb{M}_d(X\mathbf{m}) \succeq 0 \tag{3b}$$

$$\textstyle\sum_\alpha a_{j\alpha} \mathbf{m}_\alpha = b_j \qquad \forall j = 1, \ldots, m. \tag{3c}$$

The sequence of upper bounds $p_d^* \geq p_{d+1}^* \geq \ldots \geq p^*$ will converge as $\lim_{d \to \infty} p_d^* = p^*$ if X is Archimedean. Given that the per-iteration complexity of an Interior Point SDP solver in with a M affine constraints involving PSD matrix constraint of size N is $O(N^3 M + N^2 M^2)$ [17] and the moment-SOS hierarchy results in $N = \binom{n+d}{d}$ (with M scaling in a polynomial manner based on (n, d)), the computational cost of calculating p_d^* from (3a) will therefore increase rapidly as both n and d grow.

D. Occupation Measures for Peak Estimation

The Ordinary Differential Equation (ODE) peak estimation problem finds the maximum value of a function $p(x)$ along system trajectories,

$$P^* = \max_{t \in [0,T],\, x_0 \in X_0} p(x(t \mid x_0)), \quad \dot{x}(t') = f(t', x(t')). \tag{4}$$

Optimizing trajectories of (4) may be represented by (x_0^*, t_p^*, x_p^*) that satisfies $P^* = p(x_p^*) = p(x(t_p^* \mid x_0^*))$.

Figure 1 performs a peak estimation problem in times $t \in [0, 5]$ for the Flow system from [8],

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 - x_2 + \frac{1}{3} x_1^3 \end{bmatrix}. \tag{5}$$

Trajectories starting from $X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$ in the black circle are drawn in cyan. The minimal vertical coordinate $\min x_2$ is $P^* = -0.5734$, and its optimal trajectory in dark blue takes place in $x_0^* \approx (1.4889, -0.3998)$ (blue circle), $x_p^* \approx (0.6767, -0.5734))$ and time $t_p^* \approx 1.6627$.

ODE peak estimation may be solved through a primal-dual pair of LPs. The measure LP in (10) involves an occupation measure $\mu \in \mathcal{M}_+([0, T] \times X)$, an initial measure $\mu_0 \in \mathcal{M}_+(X_0)$, a peak measure $\mu_p \in \mathcal{M}_+([0, T] \times X)$. The occupation measure $\mu(A \times B)$ for $A \subseteq [0, T], B \subseteq X$ given

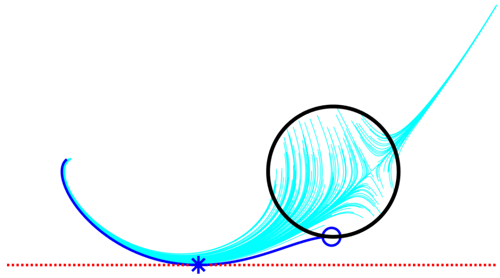


Fig. 1: Minimizing x_2 on the Flow system (5)

a measure μ_0 over initial conditions and a stopping time $t^* \in [0, T]$ is,

$$\mu(A \times B) = \int_{[0,t^*]\times X_0} I_{A\times B}\left((t, x(t \mid x_0))\right) dt\, d\mu_0(x_0). \tag{6}$$

Definition (6) may be interpreted that $\mu(A \times B)$ is the average amount of time a trajectory with initial condition drawn according μ_0 spends in the box $A \times B$. The Lie derivative along acODE dynamics $\dot{x} = f(t, x)$ of a function $v \in C^1([0, T] \times X)$ is,

$$\mathcal{L}_f v(t, x) = \partial_t v(t, x) + f(t, x) \cdot \nabla_x v(t, x). \tag{7}$$

The measures (μ_0, μ_p, μ) are connected by Liouville's Equation, Liouville's equation expresses the constraint that μ_0 is connected to μ_p by trajectories with dynamics f for all test functions $v \in C^1([0, T] \times X)$,

$$\langle v(t, x), \mu_p\rangle = \langle v(0, x), \mu_0\rangle + \langle \mathcal{L}_f v(t, x), \mu\rangle \tag{8}$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu, \tag{9}$$

in which (9) is equivalent to (8) holding for all v. The measure LP for peak estimation from [5] is,

$$\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p\rangle & \text{(10a)}\\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu & \text{(10b)}\\
& \langle 1, \mu_0\rangle = 1 & \text{(10c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) & \text{(10d)}\\
& \mu_0 \in \mathcal{M}_+(X_0), & \text{(10e)}
\end{aligned}$$

and its dual in terms of variables (v, γ) is,

$$\begin{aligned}
d^* = \min_{v(t,x),\gamma} \quad & \gamma & & \text{(11a)}\\
& \gamma \geq v(0, x) & \forall x \in X_0 \quad & \text{(11b)}\\
& \mathcal{L}_f v(t, x) \leq 0 & \forall (t, x) \in [0, T] \times X \quad & \text{(11c)}\\
& v(t, x) \geq p(x) & \forall (t, x) \in [0, T] \times X. \quad & \text{(11d)}
\end{aligned}$$

Programs (10) and (11) satisfy strong duality with $p^* = d^*$ under mild conditions, and the bound $p^* \geq P^*$ from (4) and (10) will be tight when $[0, T] \times X$ is a compact set [1], [5]. The moment-SOS hierarchy has been employed to find a convergent sequence of upper bounds to program (10) [6]. Near-optimal trajectories may be localized through sublevel sets [6] approximately recovered if the obtained moment matrices obey rank conditions [14].

III. Distance Program

A trajectory that achieves a minimal distance of closest approach to X_u (is an optimal solution to Program (1)) can be represented by a tuple $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$ as defined in Table I below.

TABLE I: Representation of distance-minimizing trajectory

x_p^*	point of closest approach on trajectory
y^*	point of closest approach on unsafe set
x_0^*	initial condition generating x_p^*
t_p^*	time needed to travel from x_0^* to x_p^*

The relationship between these quantities for an optimal trajectory of (1) is:

$$P^* = c(x_p^*; X_u) = c(x_p^*, y^*) = c(x(t_p^* \mid x_0^*), y^*). \tag{12}$$

Figure 2 plots a the result of an L_2 distance estimation problem between the Flow system (5) and the half-circle unsafe set $X_u = \{x \in \mathbb{R}^2 \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2,\ \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$. This distance of closest L_2 approach is 0.2831. The red curve marks the level set of all points that are this optimal distance away from X_u. The optimizing trajectory starts at $x_0^* \approx (1.489, -0.3998)$ (blue circle) and reaches a minimal distance at time $t^* \approx 0.6180$ at the point $x_p^* \approx (0, -0.2997)$ (blue star). The corresponding closest point on X_u is $y^* \approx (-0.2002, -0.4998)$ (blue square).

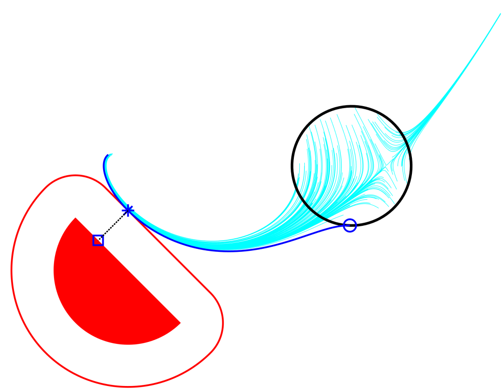


Fig. 2: Flow system trajectories remain at least an L_2 bound of 0.2831 away from X_u

A. Assumptions

The assumptions placed on distance program (1) are,

- A1 T is finite and the set X is compact
- A2 The dynamics function $f(t, x)$ is Lipschitz
- A3 The cost $c(x, y)$ is a member of $C^0(X \times X_u)$.
- A4 Any trajectory with $x(t \mid x_0) \notin X$ for $x_0 \in X_0 \subset X,\ t \in [0, T]$ also satisfies $x(t' \mid x_0) \notin X\ \forall t' \in [t, T]$ (non-return)

B. Measure Program

Theorem 3.1: The following infinite-dimensional LP in measure variables $(\mu_0, \mu_p, \mu, \eta)$ will lower bound Program

(1) under assumptions A1-A4,

$$p^* = \inf \quad \langle c(x,y), \eta \rangle \tag{13a}$$
$$\pi^x_\# \eta = \pi^x_\# \mu_p \tag{13b}$$
$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{13c}$$
$$\langle 1, \mu_0 \rangle = 1 \tag{13d}$$
$$\eta \in \mathcal{M}_+(X \times X_u) \tag{13e}$$
$$\mu_p, \ \mu \in \mathcal{M}_+([0,T] \times X) \tag{13f}$$
$$\mu_0 \in \mathcal{M}_+(X_0). \tag{13g}$$

Proof: Assume that $(x_p^*, y^*, x_0^*, t_p^*)$ is a representation of an optimal trajectory satisfying relation (12). Rank-one atomic probability measures $\mu_0^* = \delta_{x=x_0^*}$, $\mu_p^* = \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$, and $\eta^* = \delta_{x=x_p^*} \otimes \delta_{y=y^*}$ may be constructed from this trajectory. An occupation measure μ^* which is the unique measure satisfying $\langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$ for all $v(t,x) \in C([0,T] \times X)$ may also be formed. The measures $(\mu_0^*, \mu_p^*, \mu^*, \eta^*)$ are a feasible solution to constraints (13b)-(13g) with an objective $\langle c, \eta^* \rangle = c(x_p^*, y^*) = P^*$. It follows that p^* is a lower bound on the feasible P^*. ■

Lemma 3.2: When A1-A4 are satisfied, all measures will have finite mass.

Proof: Constraint (13d) clamps $\langle 1, \mu_0 \rangle = 1$, which imposes through constraint (13c) with $v(t,x) = 1$ that $\langle 1, \mu_p \rangle = \langle 1, \mu_0 \rangle = 1$. Similarly, constraint (13c) ($v(t,x) = 1$) requires $\langle 1, \eta \rangle = \langle 1, \mu_p \rangle = 1$ with $w(x) = 1$. Lastly, μ will have bounded mass $\langle 1, \mu \rangle = \langle t, \mu_p \rangle < T$ by constraint (13c) with $v(t,x) = t$. ■

C. Function Program

A Lagrangian $\mathscr{L}$ associated with problem (13) possesses dual variables $v(t,x) \in C([0,T] \times X)$, $w(x) \in C(X)$, $\gamma \in \mathbb{R}$ corresponding to constraints (13b)-(13d),

$$\begin{aligned}\mathscr{L} = &\langle c(x,y), \eta \rangle + \langle v(t,x), \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu - \mu_p \rangle \\ &+ \langle w(x), \pi^x_\# \mu_p - \pi^x_\# \eta \rangle + \gamma(1 - \langle 1, \mu_0 \rangle).\end{aligned} \tag{14}$$

The dual program as obtained by taking a saddle point of the Lagrangian (14),

$$d^* = \sup_{\gamma, v, w} \ \inf \mu_0, \mu_p, \mu, \eta \mathscr{L} \tag{15a}$$
$$= \sup_{\gamma \in \mathbb{R}} \quad \gamma \tag{15b}$$
$$v(0,x) \geq \gamma \qquad \forall x \in X_0 \tag{15c}$$
$$c(x,y) \geq w(x) \qquad \forall (x,y) \in X \times X_u \tag{15d}$$
$$w(x) \geq v(t,x) \qquad \forall (t,x) \in [0,T] \times X \tag{15e}$$
$$\mathcal{L}_f v(t,x) \geq 0 \qquad \forall (t,x) \in [0,T] \times X \tag{15f}$$
$$w \in C(X) \tag{15g}$$
$$v \in C^1([0,T] \times X). \tag{15h}$$

Theorem 3.3: Problems (13) and (15) are dual to each other, and satisfy strong duality with $p^* = d^*$ when assumptions A1-A4 hold. Additionally, the infimum is attained.

Proof: A proof of strong duality and attainment is given in Appendix A the extended version of this paper [15] based on arguments from Theorem 2.6 of [18]. ■

Theorem 3.4: The solution d^* from (15) is equal to P^* from (1) if assumptions A1-A4 are satisfied.

Proof: This equality will be demonstrated by proving that $P^* - \delta \leq d^* \leq P^*$ for every $\delta > 0$, with d^*. Strong duality (Theorem 3.3) imposes that $p^* = d^*$ with $p^* \leq P^*$ (Theorem 3.1). This implies that $d^* \leq P^*$.

To address the lower bound $P^* - \delta \leq d^*$, a feasible tuple (γ, v, w) for problem (15) must be generated with value $\gamma = P^* - \delta$. By assumption A3, w may be chosen as the C^0 function $c(x; X_u)$. A v may be constructed using Appendix D of [6], in which a function $W \in C^1([0,T] \times X)$ may be found satisfying the following equations (D.2 and D.3 from citefantuzzi2020bounding with a minimization objective)

$$\mathcal{L}_f W(t,x) \geq -\delta/(5T) \qquad \forall (t,x) \in [0,T] \times X \tag{16a}$$
$$w(x) \geq W(t,x) - (2/5)\delta \qquad \forall (t,x) \in [0,T] \times X \tag{16b}$$
$$W(0,x) \geq \gamma \qquad \forall x \in X_0 \tag{16c}$$
$$\gamma \geq P^* - (2/5)\delta. \tag{16d}$$

v may be chosen using W as,

$$v(t,x) = W(t,x) - (2/5)\delta - \delta/(5T)(T-t). \tag{17}$$

The functon W is constructing using the trajectory flow map for dynamics f (Lemma D.2 of [6]), producing a valid tuple (γ, v, w) for (15) with $\gamma = P^* - \delta$ and proving that $P^* - \delta \leq d^* \leq P^*$. ■

Remark 1: A chain of lower bounds may be found $v(t,x) \leq w(x) \leq c(x; X_u)$ holding $\forall (t,x) \in [0,T] \times X$ for all $\forall (t,x) \in [0,T] \times X$.

D. LMI Program

The moment-SOS hierarchy may be used to approximate program (13) from below in the case where $f(t,x)$ and $c(t,x)$ are polynomial and the sets (X_0, X, X_u) each are basic semialgebraic and Archimedean. Assume that these sets may be described by a finite number of bounded-degree polynomial inequality constraints,

$$\begin{aligned} X_0 &= \{x \in \mathbb{R}^n \mid g_k^0(x) \geq 0, \ \forall k = 1, \ldots, N_0\} \\ X &= \{x \in \mathbb{R}^n \mid g_k^X(x) \geq 0, \ \forall k = 1, \ldots, N_X\} \\ X_u &= \{x \in \mathbb{R}^n \mid g_k^U(x) \geq 0, \ \forall k = 1, \ldots, N_U\}. \end{aligned}$$

The polynomials $g_k^0(x), g_k^X(x), g_k^U(x)$ have bounded degrees d_k^0, d_k, d_k^U respectively.

The Kronecker delta tensor δ_{ij} has a value of $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Passing a test function $v(t,x) = x^\alpha t^\beta$ for multi-index powers $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}$ into the Liouville equation (13c) yields the relation,

$$\langle x^\alpha, \mu_0 \rangle \delta_{\beta 0} + \langle \mathcal{L}_f(x^\alpha t^\beta), \mu \rangle - \langle x^\alpha t^\beta, \mu_p \rangle = 0. \tag{18}$$

Let $(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}, \mathbf{m}^\eta)$ be a sequence of moments of the measures $(\mu_0, \mu_p, \mu, \eta)$. The operation $\text{Liou}_{\alpha\beta}(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m})$ may be understood as the induced relation in moment sequences inspired by (18). Define the dynamics degree $\tilde{d}$ as $\tilde{d} = d - 1 + \lceil \deg(f)/2 \rceil$. The application of the moment-SOS hierarchy on the measure program (13) yields the following LMI in each degree d,

$$
\begin{aligned}
p_d^* = \min \quad & \textstyle\sum_{\alpha,\gamma} c_{\alpha\gamma} \mathbf{m}^\eta_{\alpha\gamma}. && \text{(19a)}\\
& \mathbf{m}^\eta_{\alpha 0} = \mathbf{m}^p_{\alpha 0} \qquad \forall \alpha \in \mathbb{N}^n_{\le 2d} && \text{(19b)}\\
& \mathrm{Liou}_{\alpha\beta}(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}) = 0 \quad \forall (\alpha,\beta) \in \mathbb{N}^{n+1}_{\le 2d} && \text{(19c)}\\
& \mathbf{m}^0_0 = 1 && \text{(19d)}\\
& \mathbb{M}_d(X_0 \mathbf{m}^0) \succeq 0 && \text{(19e)}\\
& \mathbb{M}_d(([0,T]\times X)\mathbf{m}^p) \succeq 0 && \text{(19f)}\\
& \mathbb{M}_{\tilde{d}}(([0,T]\times X)\mathbf{m}) \succeq 0 && \text{(19g)}\\
& \mathbb{M}_d((X \times X_u)\mathbf{m}^\eta) \succeq 0. && \text{(19h)}
\end{aligned}
$$

Theorem 3.5: The sequence of lower bounds of program (19) will converge to (1) as $\lim_{d\to\infty} p_d^* = p^* = d^*$ under assumptions A1-A4 and if (X_0, X, X_u) are each Archimedean.

Proof: This result holds through the use of Lemma 3.2, the Archimedean assumption, and Corollary 8 of [19]. ∎

Table II lists the sizes of the moment matrices (PSD matrix constraints) that appear in the LMI (19). The largest PSD constraint is $\mathbb{M}_d[m^\eta] \ge 0$ with matrix size $\binom{2n+d}{d}$, except in cases where the dynamics f have a very high polynomial degree. Computational complexity of the LMI problem (19) therefore rises in a polynomial manner as n increases for each fixed degree d.

TABLE II: Dimension of Moment Matrices in (19)

Moment	$\mathbb{M}_d(m^0)$	$\mathbb{M}_d(m^p)$	$\mathbb{M}_{\tilde{d}}(m)$	$\mathbb{M}_d(m^\eta)$
Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

IV. Numerical Examples

Code to generate examples is available at https://github.com/jarmill/distance. Dependencies include Gloptipoly [20], YALMIP [21], and Mosek [22]. All examples will feature an L_2 distance objective unless indiciated otherwise. The returned bounds are the estimated L_2 norms, which are the square roots of the LMI outputs.

A. Flow Moon

The Flow example in Figure 2 features a convex set X_u. The unsafe set in Figure 3 is non-convex Moon-shaped set, which is formed by the region inside the circle centered at $(0.4, -0.4)$ with radius 0.8 and outside the circle centered at $(0.6596, 0.3989)$ with radius 1.16.

L_2 distance bounds for degrees $1 : 5$ are $L_2^{1:5} = [1.487\times 10^{-4}, 2.433\times 10^{-4}, 0.1501, 0.1592, 0.1592]$. Figure 3 pictures the degree 5 LMI bound. A near-optimal trajectory of $x_0^* \approx (1.489, -0.3998)$, $x_p^* \approx (1.113, -0.4956)$, $y^* \approx (1.161, -0.6472)$ and $t_p^* \approx 0.1727$ was recovered because the moment matrices $\mathbb{M}_5(m^0)$, $\mathbb{M}_5(m^p), \mathbb{M}_5(m^\eta)$ were rank-1 up to numerical accuracy.

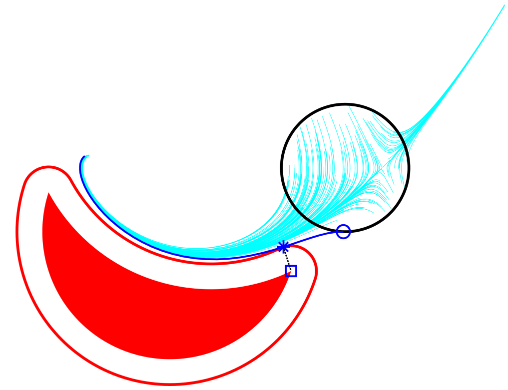


Fig. 3: L_2 bound of 0.1592

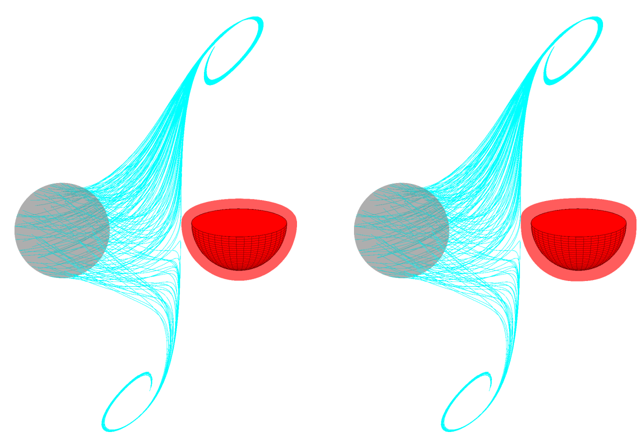
B. Twist

The three-state Twist dynamical system has parameter matrices A and B,

$$\dot{x}_i(t) = \textstyle\sum_j A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2, \tag{20}$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{21}$$

Trajectories in Figure 4 begin in the gray sphere $X_0 = \{x \mid (x_1+0.5)^2 + x_2^2 + x_3^2 \le 0.2^2$ and run until time $T = 5$. The unsafe set is the red half-sphere $X_u = \{x \mid (x_1 - 0.25)^2 + x_2^2 + x_3^2 \le 0.2^2,\ x_3 \le 0\}$. The red shell surrounding X_u are distance contours found through the degree-5 relaxation of LMI program (19). The L_2 distance in Figure 4a has bounds $L_2^{1:5} = [0, 0, 0.0336, 0.0425, 0.0427]$, and the L_4 distance in Figure 4b yields bounds $L_4^{2:5} = [0, 0.0298, 0.0408, 0.0413]$.



(a) L_2 bound of 0.0427 (b) L_4 bound of 0.0411

Fig. 4: Twist system trajectories and degree-5 distance bound sublevel sets (20)

V. Extension to Dynamic Uncertainty

ODE Peak estimation problems were extended to systems with uncertainty in [23], and this section will demonstrate how distance estimation problems may be solved for systems with uncertainty. Let $h : [0, T] \to H$ be a Borel measurable uncertainty process that may vary arbitrarily quickly in time taking on values in a compact range $H \subset \mathbb{R}^{N_h}$. A distance estimation problem involving dynamics with a time-dependent uncertainty process $h(t)$,

$$\begin{aligned} P^* = \min_{t,\, x_0,\, y,\, h} \ & c(x(t \mid x_0, h(t)), y) \\ & \dot{x}(t') = f(t', x, h(t')) && \forall t' \in [0, T] \\ & h(t') \in H && \forall t' \in [0, T] \\ & t \in [0, T], \ x_0 \in X_0, \ y \in X_u. \end{aligned} \tag{22}$$

The optimal trajectory $(x_0^*, x_p^*, t_p^*, y^*, h^*(t))$ achieving a distance of P^* in problem (22) has a unique occupation measure representation $\mu_h \in \mathcal{M}_+([0, T] \times X \times H)$ of,

$$\mu_h : \quad \langle \bar{v}(t, x, h), \mu_h \rangle = \int_0^{t_p^*} \bar{v}(t, x(t \mid x_0^*, \ h^*(t)), h^*(t)) dt,$$

valid for all $\bar{v}(t, x, h) \in C([0, T] \times X \times H)$. A controlled Liouville equation to replace (13c) $\forall v \in C^1([0, T] \times X)$ is,

$$\langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_{f(t,x,h)} v(t, x), \mu_h \rangle. \tag{23}$$

An example of this uncertainty approach is in performing distance estimation on the following corrupted flow system with $h(t) \in [-0.25, 0.25] \ \forall t \in [0, T]$,

$$\dot{x} = \begin{bmatrix} x_2 \\ (-1 + h) x_1 - x_2 + \frac{1}{3} x_1^3 \end{bmatrix}. \tag{24}$$

The first five L_2 distance bounds from the LMI relaxation are $L_2^{1:5} = [5.125 \times 10^{-5}, 1.487 \times 10^{-4}, 0.1609, 0.1688, 0.1691]$. Figure 5 visualizes sampled trajectories along with a $L_2^5 = 0.1691$ distance contour.

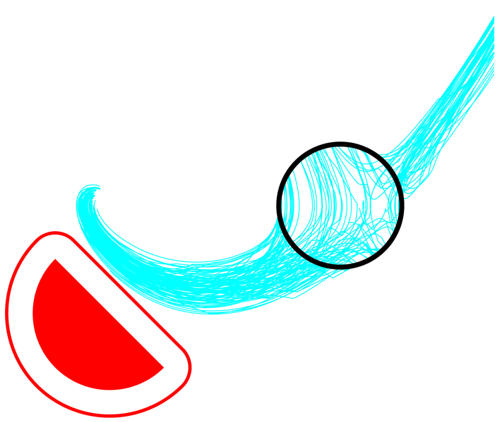


Fig. 5: Flow (24) with time-dependent uncertainty has an L_2 bound of 0.1691

VI. Conclusion

An infinite-dimensional LP in measures was developed to lower bound the distance of closest approach between points along trajectories and points on the unsafe set. The optimal value of this LP is arbitrarily close to the true minimal distance under assumptions A1-A5, and the moment-SOS hierarchy will additionally converge as the degree $d \to \infty$ under a polynomial (and Archimedean) setting. Distance estimation changes the cost structure of the occupation measure peak estimation problem, and can therefore be integrated with complementary methods to treat dynamical uncertainty. Future work involves exploiting problem structure to reduce the cost of solving LMI relaxations and creating an optimal control scheme to maximize the distance of closest approach.

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