Bounding distances to unsafe sets

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MFO: Conic Linear Optimization
for Computer-Assisted Proofs

Northeastern University

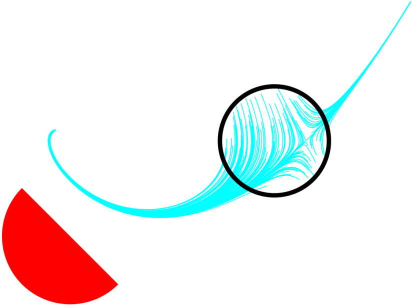
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Relax distance using optimal transport theory

Develop occupation measure programs to bound distance

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$

$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2,$$
$$\sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Distance Function

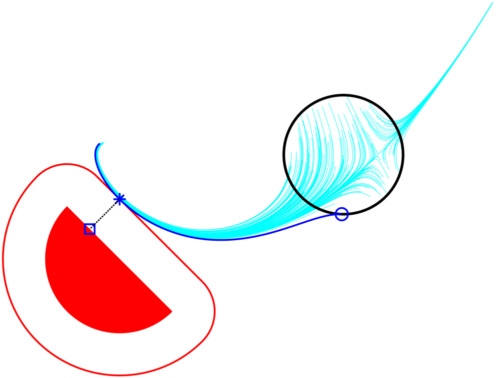
Metric space (X, c) satisfying $\forall x, y \in X$:

$$
\begin{aligned}
&c(x,y) > 0 && x \neq y \\
&c(x,x) = 0 && \\
&c(x,y) = c(y,x) && \\
&c(x,y) \leq c(x,z) + c(z,y) && \forall z \in X
\end{aligned}
$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x,y)$

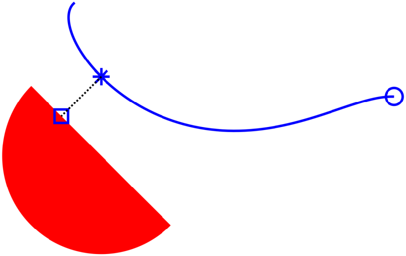
Distance Estimation Problem

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$
$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$:

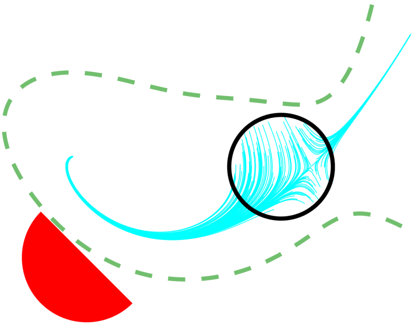
x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Safety Background

Barrier Program

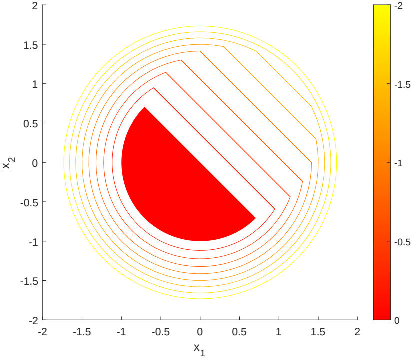
Barrier function $B : X \to \mathbb{R}$ indicates safety

$$
\begin{aligned}
&B(x) \leq 0 && \forall x \in X_u \\
&B(x) > 0 && \forall x \in X_0 \\
&f(x) \cdot \frac{\partial B}{\partial x}(x) \geq 0 && \forall x \in X
\end{aligned}
$$



Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq -x_1 - x_2$$

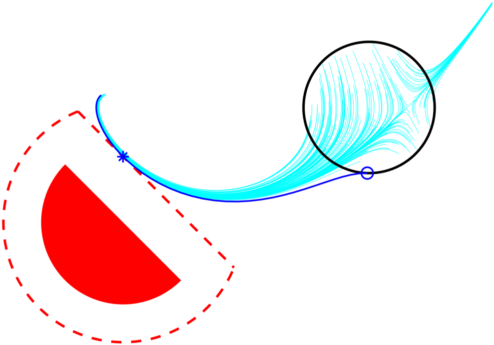
$$q = -0.25, -0.5, \ldots, -2$$

Safety Margin

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \; \forall i = 1 \dots N_u\}$

Safety margin $p^* = \max \min_i p_i(x)$ along trajectories

If $p^* < 0$, no trajectories enter X_u (safe)

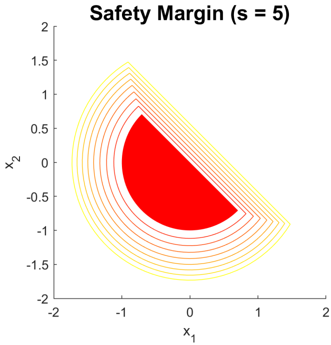


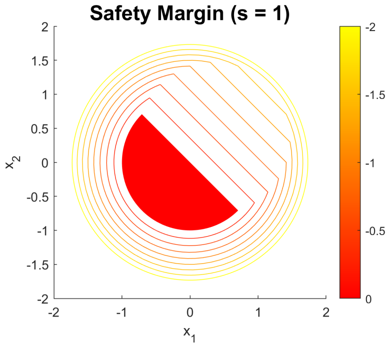
safe: $p^* \leq -0.2831$

Safety Margin Scaling

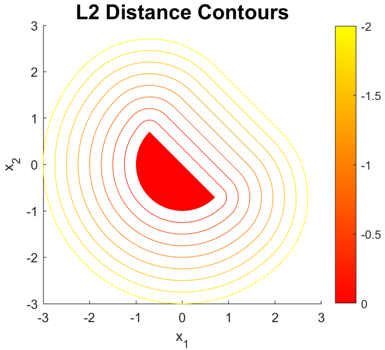
Scale factor in constraints

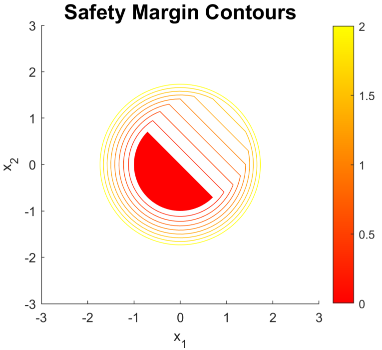
$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq s(-x_1 - x_2)$$





Distance vs. Safety Margin





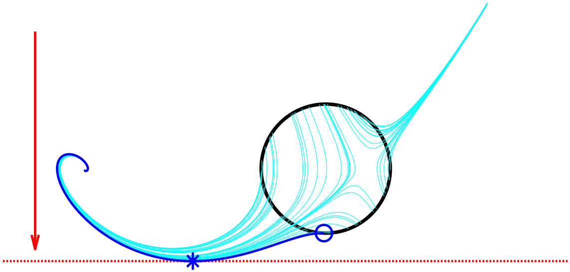
Peak Estimation

Peak Estimation Background

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

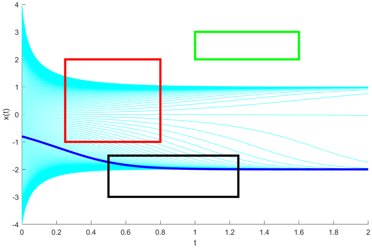
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

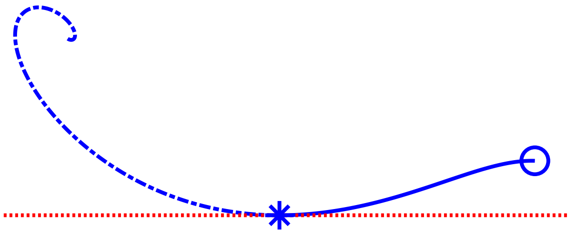
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\begin{aligned}
\mu_0^* &: \quad \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* &: \quad \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* &: \quad \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

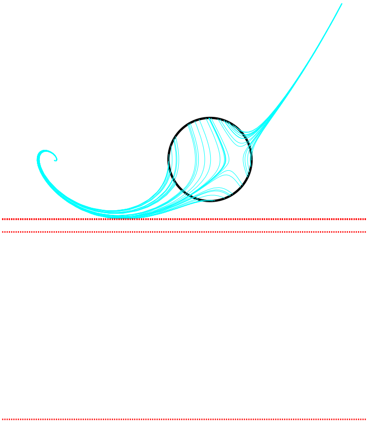
$$
\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p \rangle && \text{(1a)} \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1b)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(1c)} \\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(1d)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



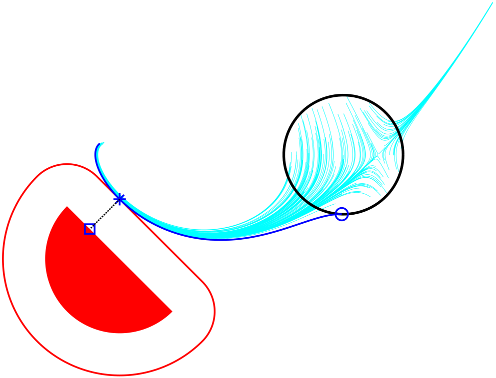
Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Distance Relaxation

Distance in points $\rightarrow$ Earth-Mover distance

$$\begin{array}{lcl} c(x,y) & & \langle c(x,y), \eta \rangle \\ x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\ y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u) \end{array}$$

Joint (Wasserstein) probability measure η

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0,T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \langle w(x), \eta(x,y) \rangle = \langle w(x), \mu_p(t,x) \rangle & \forall w \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle & \forall v \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

Approximation and Recovery

Use moment-SOS hierarchy (Archimedean assumption)

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Attempt recovery if LMI solution has low rank

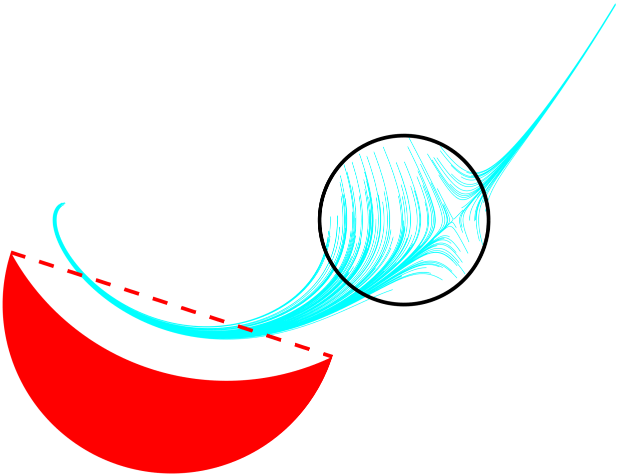
Moment matrices for (μ_0, μ_p, η) are rank-1

Related to optima extraction in polynomial optimization

Moon L2 Contours

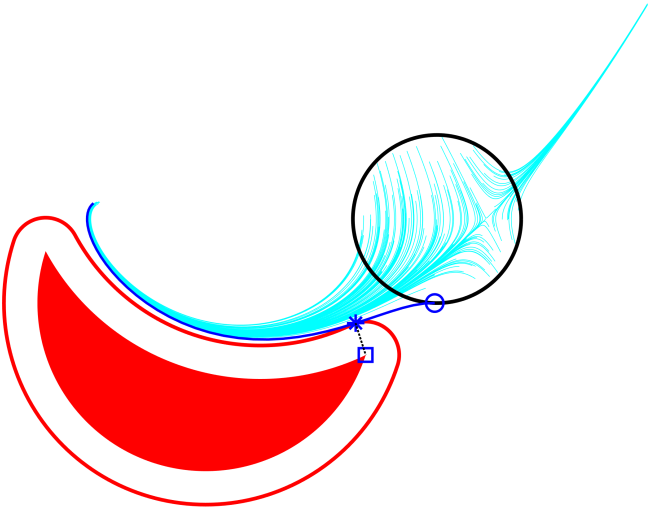
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

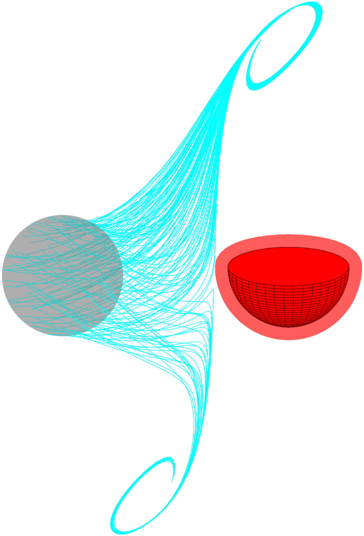
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

Distance Variations

Distance Variations

Uncertainty in dynamics

Lifted distances (with absolute values)

Sparsity

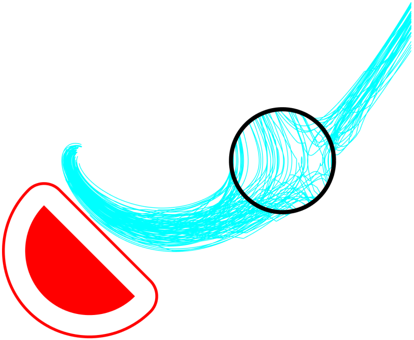
Set-Set distances for shape safety

Distance Uncertainty

Time dependent uncertainty $w(t) \in W \ \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

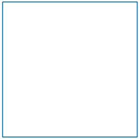
Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_{f(t,x,w)} v(t,x), \mu \rangle$



L_2 bound of 0.1691

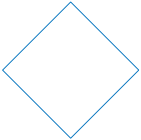
Lifted Distance

LP lifts to deal with absolute values



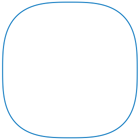
$\|x - y\|_\infty$

$$\begin{aligned} \min \quad & q \\ -q \leq \langle x_i - y_i, \eta \rangle \leq q & \qquad \forall i \end{aligned}$$



$\|x - y\|_1$

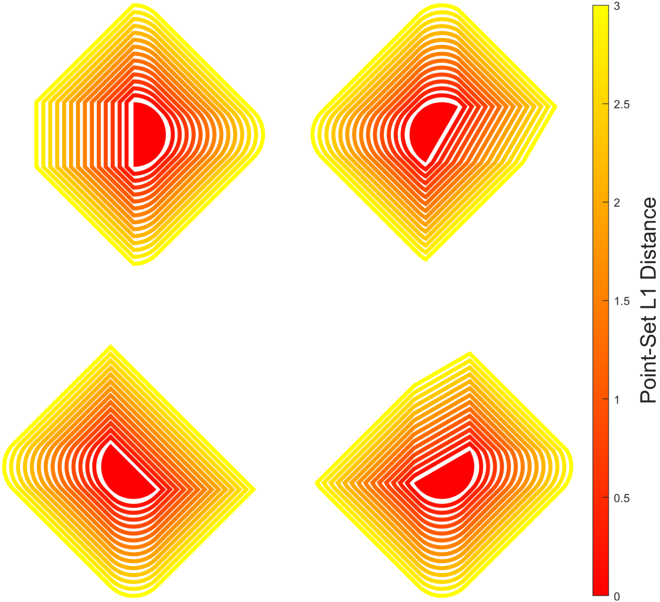
$$\begin{aligned} \min \quad & \textstyle\sum_i q_i \\ -q_i \leq \langle x_i - y_i, \eta \rangle \leq q_i & \qquad \forall i \end{aligned}$$



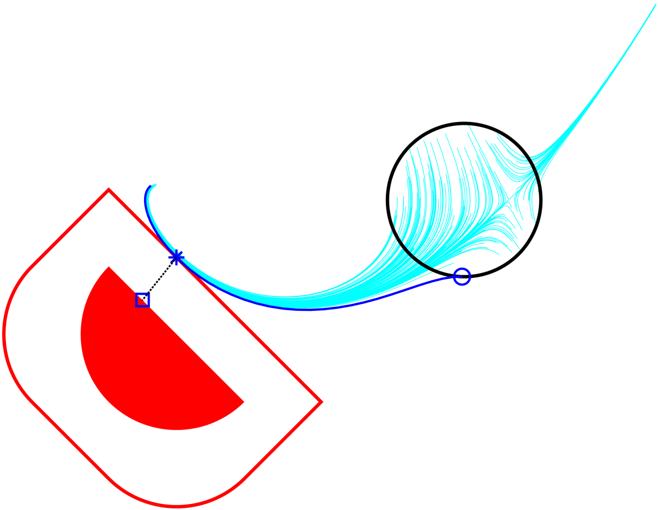
$\|x - y\|_3^3$

$$\begin{aligned} \min \quad & \textstyle\sum_i q_i \\ -q_i \leq \langle (x_i - y_i)^3, \eta \rangle \leq q_i & \quad \forall i \end{aligned}$$

Half-Circle L1 Contours



Lifted Distance (L1) Example



L_1 bound of 0.4003

Sparsity

Separable $c(x, y) = \sum_i c_i(x_i, y_i)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_k = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \dots, n$$

Sparse decomposition of η:

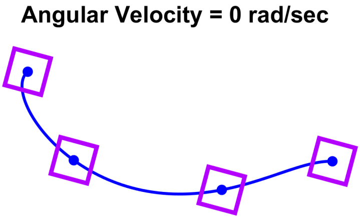
$$\begin{aligned} &\min \textstyle\sum_i \langle c_i(x_i, y_i), \eta_i \rangle && \eta^1 \in \mathcal{M}_+(X \times \mathbb{R}) \\ &\pi_\#^{I_k \cap I_{k+1}} \eta_k = \pi_\#^{I_k \cap I_{k+1}} \eta_{k+1} && \eta^k \in \mathcal{M}_+(\mathbb{R}^{n+1}) \\ &\pi_\#^x \mu^p = \pi_\#^x \eta_1 && \eta^n \in \mathcal{M}_+(\mathbb{R} \times X_u) \end{aligned}$$

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

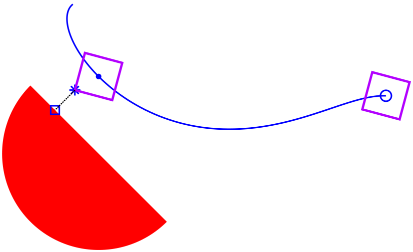
$$A : S \times \Omega \to X \qquad (s, \omega) \mapsto A(s; \omega)$$



Set-Set Distance Problem

Set-Set distance between $A(\,\cdot\,;\omega) \circ S$ and X_u given ω

$$
\begin{aligned}
P^* = \min_{t,\, \omega_0 \in \Omega_0,\, s \in S} \; & c(x(t); X_u) \\
& x(t) = A(s;\, \omega(t \mid \omega_0)) && \forall t \in [0, T] \\
& \dot{\omega}(t) = f(t, \omega) && \forall t \in [0, T]
\end{aligned}
$$



L_2 bound of 0.1465

Set-Set Program (Measures)

Add new 'shape' measure μ_s

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x, y), \eta \rangle \\
& \langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle & \forall v \\
& \langle w(x), \eta(x, y) \rangle = \langle w(A(s; \omega)), \mu_s(s, \omega) \rangle & \forall w \\
& \langle z(\omega), \mu_p(t, \omega) \rangle = \langle z(\omega), \mu_s \rangle & \forall z \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_s \in \mathcal{M}_+(\Omega \times S) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times \Omega) \\
& \mu_0 \in \mathcal{M}_+(\Omega_0)
\end{aligned}
$$

Take-aways

Conclusion

Distance Estimation with occupation measures

Approximate recovery if moment matrices are low-rank

Extend to uncertain, lifted, set-set scenarios

Future Work

- Distance-Maximizing Control
- Further Sparsity
- Efficient Computation
- Other nonnegativity cones and proofs

Acknowledgements

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`

Graduating May 2023, looking for postdocs

Bonus Material and Ideas

Distance Program (Functions)

Auxiliary $v(t,x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t,x) && \forall (t,x) \in [0,T] \times X \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& \mathcal{L}_f v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X \\
& v \in C^1([0,T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Chain $\forall (t,x,y) \in [0,T] \times X \times X_u: \;\; c(x,y) \geq w(x) \geq v(t,x)$

Lifted Distance Program (Measure)

New terms for lifted distance

$$
\begin{aligned}
p^* = \ \min \quad & \textstyle\sum_i q_i \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \langle 1, \mu_0 \rangle = 1 \\
& -q_i \leq \langle c_{ij}(x, y), \eta \rangle \leq q_i & \forall i, j \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Same process as maximin peak

Lifted Distance Program (Function)

New terms $\beta_i^{\pm}$ on costs

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t, x) && \forall (t, x) \in [0, T] \times X \\
& \textstyle\sum_{i,j} (\beta_{ij}^{+} - \beta_{ij}) c_{ij}(x, y) \geq w(x) && \forall (x, y) \in X \times X_u \\
& \mathcal{L}_f v(t, x) \geq 0 && \forall (t, x) \in [0, T] \times X \\
& 1^T (\beta_i^{+} + \beta_i^{-}) = 1, \ \beta_i^{\pm} \in \mathbb{R}_+^{n_i} && \forall i \\
& v \in C^1([0, T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Set-Set Program (Function)

Set-Set distance proxy $z(\omega) \le \max_{s \in S} c(A(s;\omega); X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,\omega) \ge \gamma && \forall x \in \Omega_0 \\
& c(x,y) \ge w(x) && \forall (x,y) \in X \times X_u \\
& w(A(s;\omega)) \ge z(\omega) && \forall (s,\omega) \in S \times \Omega \\
& z(\omega) \ge v(t,\omega) && \forall (t,\omega) \in [0,T] \times \Omega \\
& \mathcal{L}_f v(t,\omega) \ge 0 && \forall (t,\omega) \in [0,T] \times \Omega \\
& v \in C^1([0,T] \times X) \\
& w \in C(X), \ z \in C(\Omega)
\end{aligned}
$$