Bounding the Distance of Closest Approach to Unsafe Sets with Occupation Measures

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Safety Example

100 kph
Safety Example

100 kph

SAFE

100 kph
Safety Example

10 cm

100 kph
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Create linear program to bound distance

Solve using Semidefinite Programming
Flow System Setting

\[ \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \quad \forall t \in [0, 5] \]

\[ X_0 = \{ x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2 \} \]

\[ X_u = \{ x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2, \quad \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0 \} \]
Barrier function $B : X \rightarrow \mathbb{R}$ indicates safety

$B(x) \leq 0 \quad \forall x \in X_u$

$B(x) > 0 \quad \forall x \in X_0$

$f(x) \cdot \frac{\partial B}{\partial x}(x) \geq 0 \quad \forall x \in X$
Distance Function

Metric space \((X, c)\) satisfying \(\forall x, y \in X:\)

\[
\begin{align*}
c(x, y) &> 0 \quad \text{if } x \neq y \\
c(x, x) &= 0 \\
c(x, y) &= c(y, x) \\
c(x, y) &\leq c(x, z) + c(z, y) \quad \forall z \in X
\end{align*}
\]

Point-Unsafe Set distance: \(c(x; X_u) = \min_{y \in X_u} c(x, y)\)
Distance Estimation Problem (Nonconvex)

\[ P^* = \min_{t, x_0 \in X_0} c(x(t \mid x_0); X_u) \]

\[ \dot{x}(t) = f(t, x(t)) \quad \forall t \in [0, T], \quad x(0) = x_0. \]

\[ L_2 \text{ bound of } 0.2831 \]
Optimal trajectories described by $(x^*_p, y^*, x^*_0, t^*_p)$:

- $x^*_p$ location on trajectory of closest approach
- $y^*$ location on unsafe set of closest approach
- $x^*_0$ initial condition to produce $x^*_p$
- $t^*_p$ time to reach $x^*_p$ from $x^*_0$
Peak Estimation
Peak Estimation Background

Find minimum value of $p(x)$ along trajectories

$$P^* = \min_{t, \, x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \quad \forall t \in [0, T], \quad x(0) = x_0.$$
Occupation Measure

Time trajectories spend in set

Test function
\( v(t, x) \in C([0, T] \times X) \)

Single trajectory:
\[
\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0)) \, dt
\]

Averaged trajectory: \( \langle v, \mu \rangle = \int_X \left( \int_0^T v(t, x) \, dt \right) \, d\mu_0(x) \)

\[ x' = -x(x + 2)(x - 1) \]
Measures: Initial $\mu_0$, Peak $\mu_p$, Occupation $\mu$

For all functions $v(t, x) \in C([0, T] \times X)$

\[
\begin{align*}
\mu_0^* : & \quad \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : & \quad \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : & \quad \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t | x_0^*)) dt
\end{align*}
\]
Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

\[ p^* = \min \langle p(x), \mu_p \rangle \]  \hspace{1cm} (1a)

\[ \langle 1, \mu_0 \rangle = 1 \]  \hspace{1cm} (1b)

\[ \langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle \quad \forall v \]  \hspace{1cm} (1c)

\[ \mu, \mu_p \in \mathcal{M}_+(\mathbb{R}_+ \times X) \]  \hspace{1cm} (1d)

\[ \mu_0 \in \mathcal{M}_+(X_0) \]  \hspace{1cm} (1e)

Test functions \( v(t, x) \in C^1(\mathbb{R}_+ \times X) \)

Lie derivative \( \mathcal{L}_f v = \partial_t v(t, x) + f(t, x) \cdot \nabla_x v(t, x) \)

\((\mu_0^*, \mu_p^*, \mu^*)\) is feasible with \( P^* = \langle p(x), \mu_p^* \rangle \)
Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$
Distance Program
Distance Estimation Problem (reprise)

\[ P^* = \min_{t, x_0 \in X_0} c(x(t \mid x_0); X_u) \]

\[ \dot{x}(t) = f(t, x(t)) \quad \forall t \in [0, T], \quad x(0) = x_0. \]

\( L_2 \) bound of 0.2831
Connection to Peak Estimation

Specific form of problem

\[ p(x) = c(x; X_u) \]

Moment-SOS hierarchy requires polynomial data

Function \( c(x; X_u) \) generally non-polynomial

\[
\min_{y \in [-1, 1]} \| x - y \|_2 = \begin{cases} 
0 & x \in [-1, 1] \\
|x - \text{sign}(x)| & \text{else}
\end{cases}
\]
Distance Relaxation

Distance in points $\rightarrow$ Expectation of distance

$$c(x, y) \quad \langle c(x, y), \eta \rangle$$

$$x \in X \quad \rightarrow \quad \langle 1, \eta \rangle = 1$$

$$y \in X_\mu \quad \eta \in \mathcal{M}_+(X \times X_\mu)$$

Joint probability measure $\eta$

Inspired by Optimal Transport
Measures from Optimal Trajectories

Form measures from each \((x_p^*, x_0^*, t_p^*, y^*)\)

Atomic Measures (rank-1)

\[
\mu^*_0 : \quad \delta_{x=x_0^*} \\
\mu^*_p : \quad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*} \\
\eta^*_p : \quad \delta_{x=x_p^*} \otimes \delta_{y=y^*}
\]

Occupation Measure \(\forall \nu(t, x) \in C([0, T] \times X)\)

\[
\mu^* : \quad \langle \nu(t, x), \mu \rangle = \int_0^{t_p^*} \nu(t, x^*(t \mid x_0^*)) dt
\]
Infinite Dimensional Linear Program (Convergent)

\[ p^* = \min \langle c(x, y), \eta \rangle \]  \hspace{1cm} (2a)

\[ \langle 1, \mu_0 \rangle = 1 \]  \hspace{1cm} (2b)

\[ \langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle \quad \forall v \]  \hspace{1cm} (2c)

\[ \langle w(x), \eta(x, y) \rangle = \langle w(x), \mu_p(t, x) \rangle \quad \forall w \]  \hspace{1cm} (2d)

\[ \eta \in \mathcal{M}_+(X \times X_u) \]  \hspace{1cm} (2e)

\[ \mu_p, \mu \in \mathcal{M}_+([0, T] \times X) \]  \hspace{1cm} (2f)

\[ \mu_0 \in \mathcal{M}_+(X_0) \]  \hspace{1cm} (2g)

Prob. Measures: \[ \langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1 \]
Use moment-SOS hierarchy (Archimedean assumption)

Degree $d$, dynamics degree $\tilde{d} = d + \lfloor \deg(f)/2 \rfloor - 1$

Bounds: $p_d^* \leq p_{d+1}^* \leq \ldots \leq p^* = P^*$

Measure $\mu_0(x)$ $\mu_p(t, x)$ $\mu(t, x)$ $\eta(x, y)$

PSD Size $\binom{n+d}{d}$ $\binom{1+n+d}{d}$ $\binom{1+n+\tilde{d}}{\tilde{d}}$ $\binom{2n+d}{d}$

Timing scales approximately as $\max((1 + n)^6\tilde{d}, (2n)^6d)$
Attempt recovery if LMI solution has low rank
Moment matrices for \((\mu_0, \mu_p, \eta)\) are rank-1
Related to optima extraction in polynomial optimization

\[ L_2 \text{ bound of 0.2831} \]
Moon L2 Contours

Inside one circle, outside another
Collision if $X_u$ was a half-circle
Distance Example (Flow Moon)

$L_2$ bound of 0.1592
Distance Example (Twist)

‘Twist’ System, \( T = 5 \)

\[
\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2
\]

\[
A = \begin{bmatrix}
-1 & 1 & 1 \\
-1 & 0 & -1 \\
0 & 1 & -2
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-1 & 0 & -1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

\( L_2 \) bound of 0.0425
Distance Variations
Distance Uncertainty

Time dependent (bounded) uncertainty $w(t) \in W \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

Young measure $\mu(t, x, w)$, Liouville term $\langle L_f v(t, x, w), \mu \rangle$

$L_2$ bound of 0.1691, $w(t) \in [-1, 1]$
Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape $S$

Body to global coordinate transformation $A$:

$$A : S \times \Omega \to X$$

$$(s, \omega) \mapsto A(s; \omega)$$

Angular Velocity = 0 rad/sec    Angular Velocity = 1 rad/sec

Figure 1: Shape translating and (possibly) rotating
Set-Set Distance Problem

Set-Set distance between $A(S ; \omega(t))$ and $X_u$ given $t$

$$P^* = \min_{t, \omega_0 \in \Omega_0, s \in S} c(x(t); X_u)$$

$$x(t) = A(s; \omega(t | \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T], \quad \omega(0) = \omega_0$$

$L_2$ bound of 0.1465
Take-aways
Conclusion

Motivated Distance Estimation problem

Solved problem using occupation measures, SDP

Approximate recovery if moment matrices are low-rank

Extend to uncertain, set-set scenarios
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Thank you for your attention

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http://github.com/jarmill/distance