Bounding the Distance of Closest Approach to Unsafe Sets with Occupation Measures

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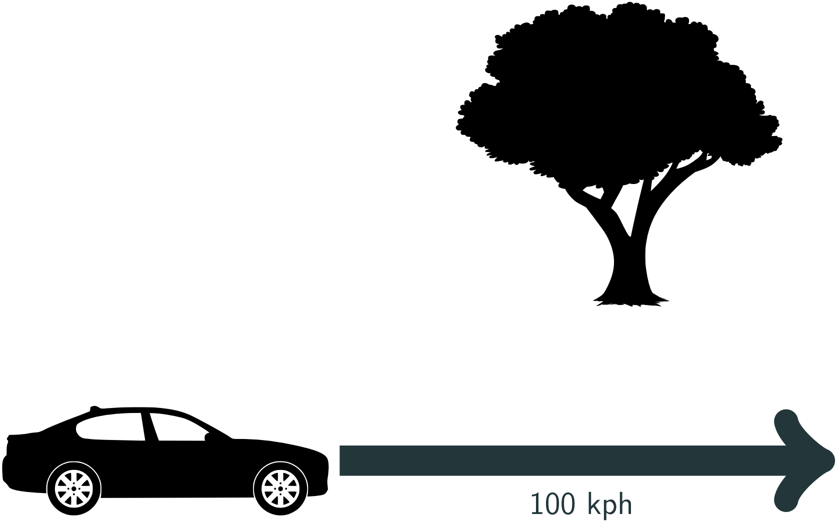
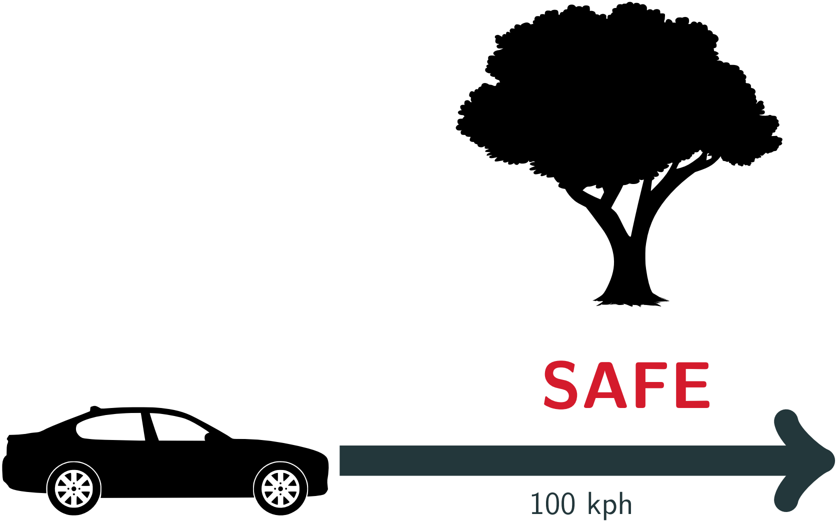
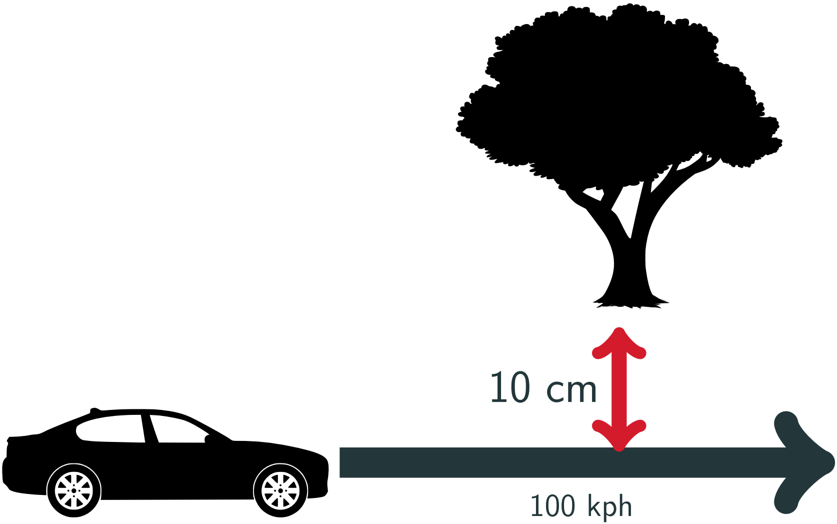
Safety Example



Safety Example



Safety Example



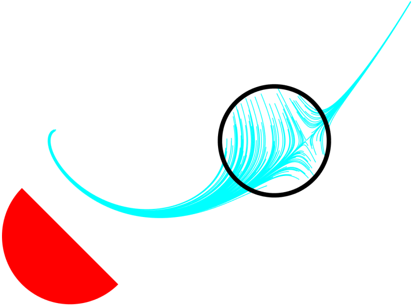
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Create linear program to bound distance

Solve using Semidefinite Programming

Flow System Setting



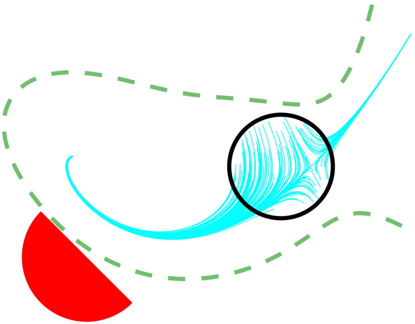
$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2,$$
$$\sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Barrier Program (Safety)

Barrier function $B : X \to \mathbb{R}$ indicates safety

$$
\begin{aligned}
B(x) &\leq 0 && \forall x \in X_u \\
B(x) &> 0 && \forall x \in X_0 \\
f(x) \cdot \frac{\partial B}{\partial x}(x) &\geq 0 && \forall x \in X
\end{aligned}
$$



Distance Function

Metric space (X, c) satisfying $\forall x, y \in X$:

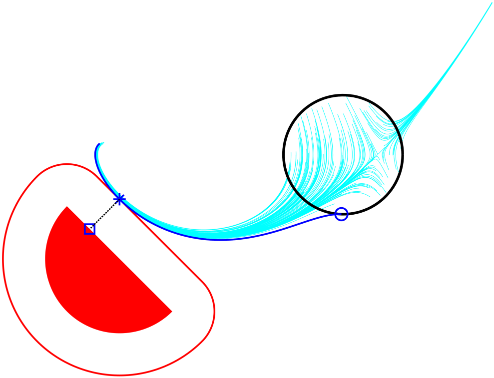
$$\begin{aligned} & c(x, y) > 0 && x \neq y \\ & c(x, x) = 0 && \\ & c(x, y) = c(y, x) && \\ & c(x, y) \leq c(x, z) + c(z, y) && \forall z \in X \end{aligned}$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x, y)$

Distance Estimation Problem (Nonconvex)

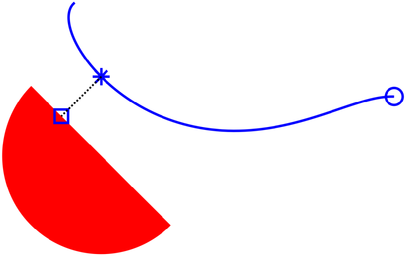
$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \quad x(0) = x_0.$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*,\ x_0^*,\ t_p^*)$:

x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

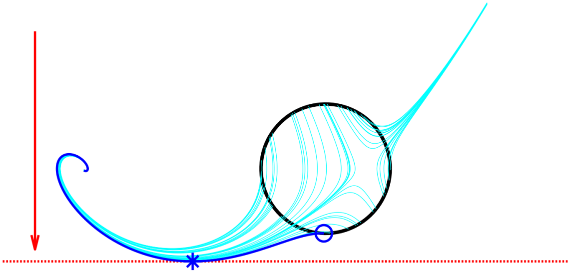
Peak Estimation

Peak Estimation Background

Find minimum value of $p(x)$ along trajectories

$$P^* = \min_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0.$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

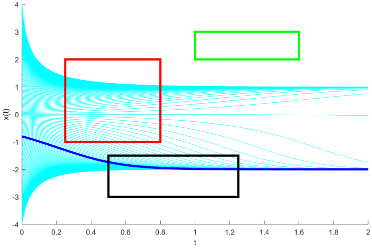
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

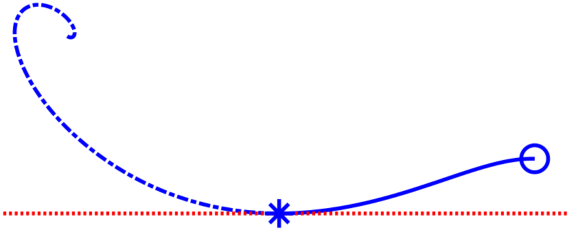
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\begin{aligned}
\mu_0^* : \qquad & \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : \qquad & \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : \qquad & \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

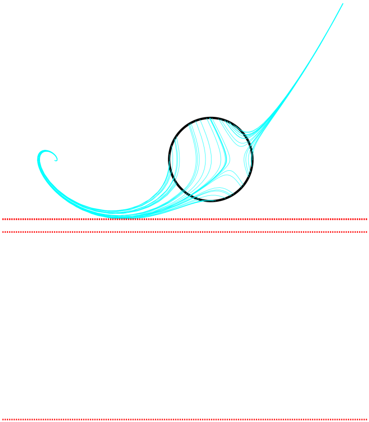
$$
\begin{aligned}
p^* = \min \quad & \langle p(x), \mu_p \rangle && \text{(1a)}\\
& \langle 1, \mu_0 \rangle = 1 && \text{(1b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(1d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)

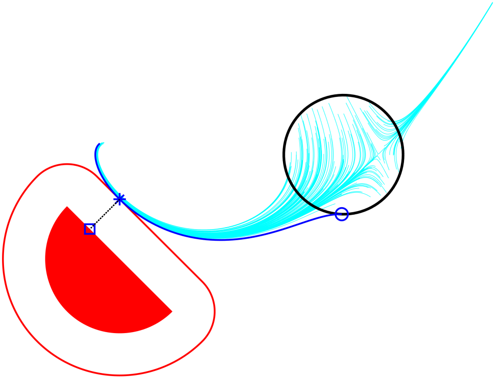
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \quad x(0) = x_0.$$



L_2 bound of 0.2831

Connection to Peak Estimation

Specific form of problem

$$p(x) = c(x; X_u)$$

Moment-SOS hierarchy requires polynomial data

Function $c(x; X_u)$ generally non-polynomial

$$\min_{y \in [-1,1]} \|x - y\|_2 = \begin{cases} 0 & x \in [-1, 1] \\ |x - \mathrm{sign}(x)| & \text{else} \end{cases}$$

Distance Relaxation

Distance in points $\rightarrow$ Expectation of distance

$$
\begin{array}{lcl}
c(x,y) & & \langle c(x,y), \eta \rangle \\
x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\
y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u)
\end{array}
$$

Joint probability measure η

Inspired by Optimal Transport

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\begin{aligned}
\mu_0^* &: \quad \delta_{x=x_0^*} \\
\mu_p^* &: \quad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*} \\
\eta^* &: \quad \delta_{x=x_p^*} \otimes \delta_{y=y^*}
\end{aligned}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0, T] \times X)$

$$\mu^* : \quad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle && (2a)\\
& \langle 1, \mu_0 \rangle = 1 && (2b)\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \qquad \forall v && (2c)\\
& \langle w(x), \eta(x,y) \rangle = \langle w(x), \mu_p(t,x) \rangle \qquad \forall w && (2d)\\
& \eta \in \mathcal{M}_+(X \times X_u) && (2e)\\
& \mu_p, \ \mu \in \mathcal{M}_+([0,T] \times X) && (2f)\\
& \mu_0 \in \mathcal{M}_+(X_0) && (2g)
\end{aligned}
$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lceil \deg(f)/2 \rceil - 1$

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Measure	$\mu_0(x)$	$\mu_p(t,x)$	$\mu(t,x)$	$\eta(x,y)$
PSD Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

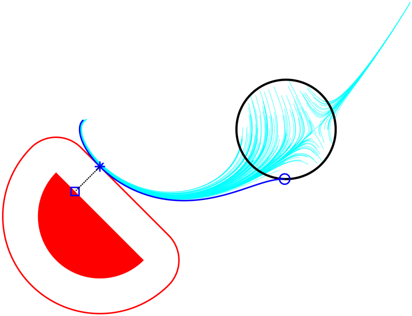
Timing scales approximately as $\max((1+n)^{6\tilde{d}}, (2n)^{6d})$

Approximation and Recovery

Attempt recovery if LMI solution has low rank

Moment matrices for (μ_0, μ_p, η) are rank-1

Related to optima extraction in polynomial optimization

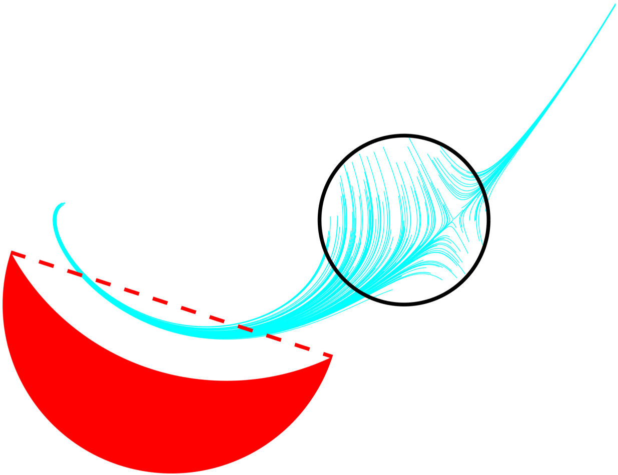


L_2 bound of 0.2831

Moon L2 Contours

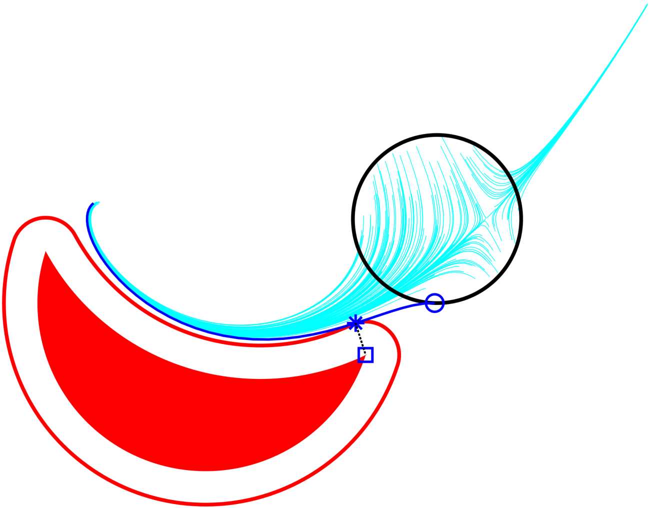
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

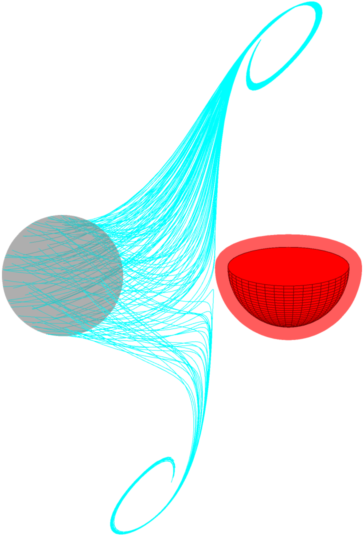
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

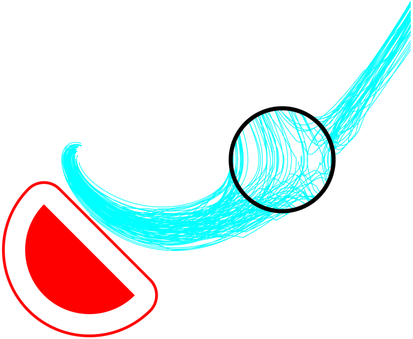
Distance Variations

Distance Uncertainty

Time dependent (bounded) uncertainty $w(t) \in W \ \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_f v(t, x, w), \mu \rangle$



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad\qquad (s, \omega) \mapsto A(s; \omega)$$

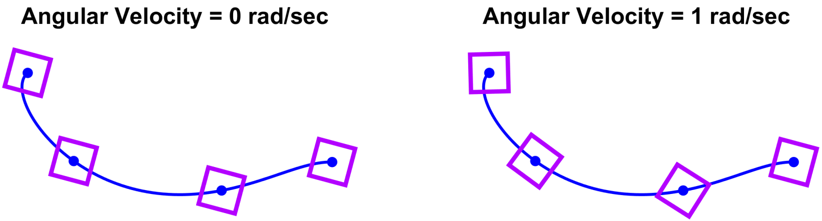


Figure 1: Shape translating and (possibly) rotating

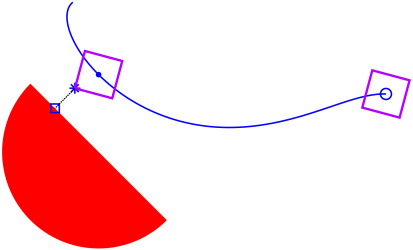
Set-Set Distance Problem

Set-Set distance between $A(S\,;\omega(t))$ and X_u given t

$$P^* = \min_{t,\, \omega_0 \in \Omega_0,\, s \in S} c(x(t); X_u)$$

$$x(t) = A(s;\, \omega(t \mid \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T], \qquad \omega(0) = \omega_0$$



L_2 bound of 0.1465

Take-aways

Conclusion

Motivated Distance Estimation problem

Solved problem using occupation measures, SDP

Approximate recovery if moment matrices are low-rank

Extend to uncertain, set-set scenarios

Acknowledgements

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`