Diameter Constrained Minimum Spanning Graphs

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Problem

Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ given

Each $e \in \mathcal{E}$ has cost c_e and distance d_e

Find minimal-cost subgraph $\mathcal{H} \subset \mathcal{G}$ under diameter budget $D_{\max}$

$$C^* = \min_{x \in \{0,1\}^{|\mathcal{E}|}} \sum_{e \in \mathcal{E}} c_e x_e \tag{1a}$$

$$\mathrm{diam}(\mathcal{H}(x)) \leq D_{\max} \tag{1b}$$

x_e: edge selection variables

Background

Bicriteria network design problem

Telecommunications: c is cost to form connection, d is transmission time

Bicriteria problems are generically non-polynomial-approximable

Prior work focuses on tree setting with uniform distances ($d_e = 1$)

Uniform trees allow for efficient precedence relations (lack of cycles)

Extensions

Directed graphs

Steiner problem (low diameter only for selected vertices)

Low-diameter communities

Multicommodity flow

Logical constraints on edge inclusions

Warm starts

Branch-and-Bound

Represent diameter as $D_{\max}$-bounded-flow between every pair of vertices (i, j) given edge weights x

Equivalent MILP to (1) is $C^* =$

$$\min_{x \in \{0,1\}^{|\mathcal{E}|}} \sum_{(k,\ell) \in \mathcal{E}} c_{k\ell}\, x_{k\ell} \tag{2a}$$

$$\forall (i,j) \in \mathcal{V} \times \mathcal{V},\ i < j : \tag{2b}$$

$$\sum_{(k,\ell) \in \mathcal{E}} d_{k\ell} f^{ij}_{k\ell} \leq D_{\max} \tag{2c}$$

$$f^{ij}_{k\ell} \leq x_{k\ell},\ (k,\ell) \in \mathcal{E} \tag{2d}$$

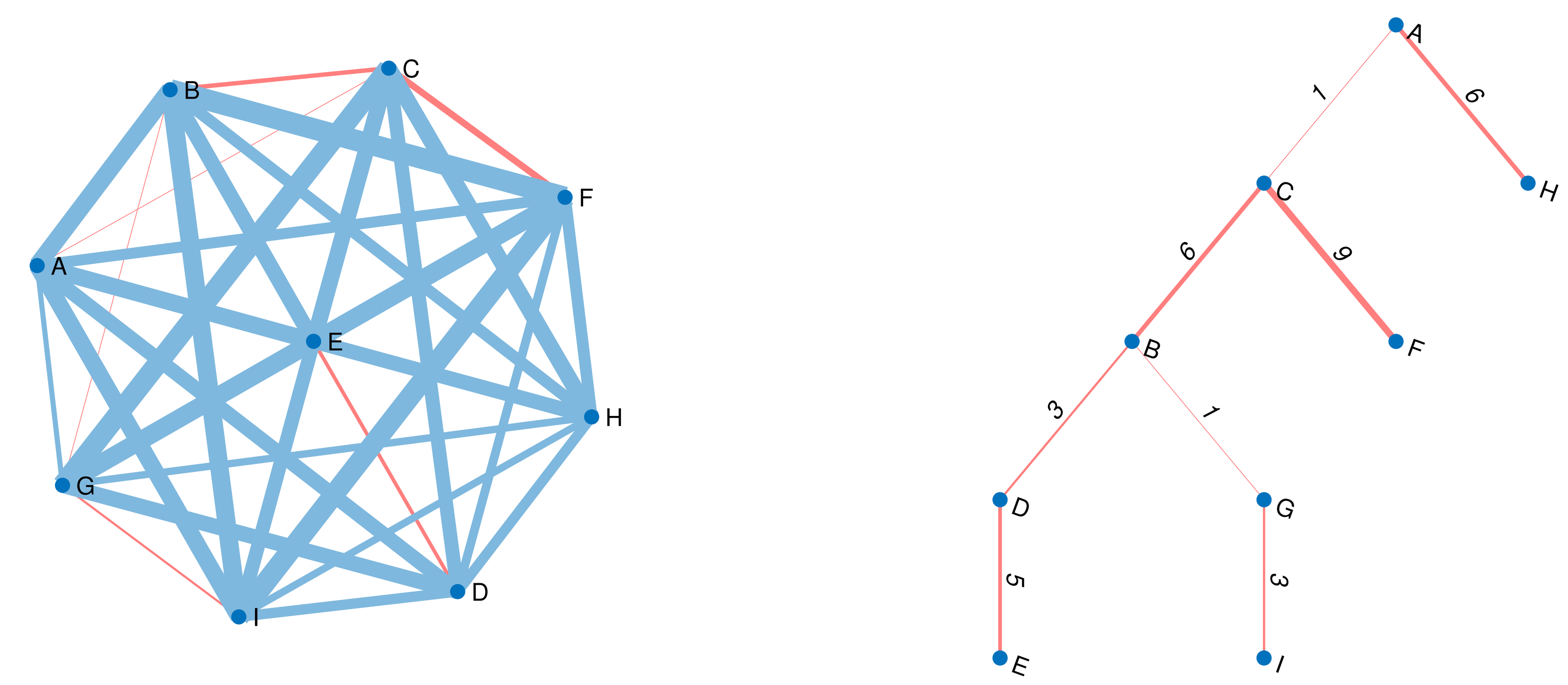
$$f^{ij}_{k\ell} = -f^{ij}_{\ell k} \tag{2e}$$

$$\sum_{\ell \in \mathcal{V}} f^{ij}_{k\ell} = \begin{cases} 1 & k \in \{i, j\} \\ 0 & \text{else}, \quad \forall k \in \mathcal{V} \end{cases} \tag{2f}$$

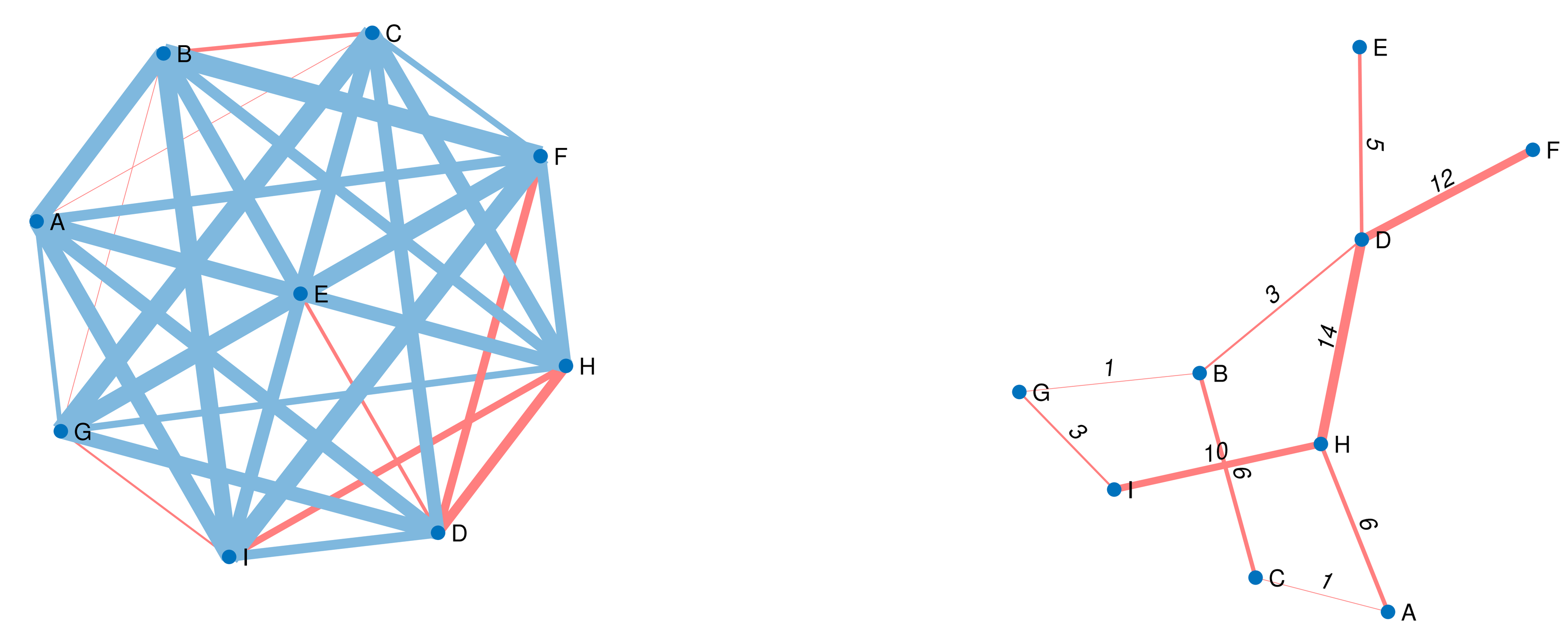
Total of $|\mathcal{E}|$ binary variables (x), $|\mathcal{E}||\mathcal{V}|(|\mathcal{V}|-1)/2$ linear variables $(f^{ij}_{k\ell})$

Example: Complete Graph

Spanning Tree: Cost 34, Diam 5



Spanning Graph: Cost 61, Diam 3



Complete graph with $|V| = 9,\ |E| = 36$

Uniform $d_e = 1$

Random costs $c_e \in 1..30$

Diameter $D_{\max} = 3$

Cycles are needed to satisfy $D_{\max}$ constraint