Data-Driven Structured Robust Control of Linear Systems

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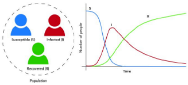
Why?

Noisy data is *collected* from system observations

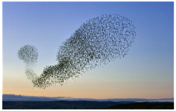
Control channels are *limited* by network structure

distributed computing

social networks

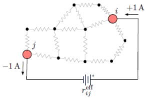


epidemic spreading



flocking of animals

traffic networks



circuits & grids

Choose $u = Kx$ to regulate performance of system

Core breakdown

Data-Driven	Synthesize K directly from data, no sysid
Structured	Require that K is in given a subspace
Robust Control	Regulate worst-case H_2 performance

of Linear Systems

Flow of Presentation

Structured Control

Set-Membership Data-Driven Control

Merge them together!

Structured Control

Structured Control

Problem: K constrained to a known *subspace* $\mathcal{S}$

Example 1: sparse control

$$K : \begin{bmatrix} \bullet & \bullet & 0 & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & 0 & \bullet & \bullet \end{bmatrix} \tag{1}$$

Example 2: sharing control ($\mathbf{1}^\top u = 0$)

$$\forall x : \mathbf{1}^\top (Kx) = 0 \quad \implies \quad \mathbf{1}^\top K = 0 \tag{2}$$

Approaches for Structured Control

Structured control is generically NP-hard [1]

Quadratic Invariance: property of graph, dynamic control [2]

We will use a convex (but conservative) scheme[3]

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[1] V. Blondel and J. N. Tsitsiklis, "NP-hardness of some linear control design problems," SIAM J. Control Optim., vol. 35, no. 6, pp. 2118– 2127, 1997

[2] L. Lessard and S. Lall, "Quadratic invariance is necessary and sufficient for convexity," in Proceedings of the 2011 American Control Conference, 2011, pp. 5360–5362.

[3] Ferrante, Francesco, Fabrizio Dabbene, and Chiara Ravazzi. "On the design of structured stabilizers for LTI systems." IEEE Control Systems Letters 4.2 (2019): 289-294.

Robust Design

H_2 norm $< \gamma$ under $u = Kx$ if program is feasible[4]:

$$\begin{aligned}
\underset{P,Q,R,L}{\text{find}} \quad & \begin{bmatrix} P - EE^\top & AR + BL \\ (AR + BL)^\top & R + R^\top - P \end{bmatrix} \succ 0 && \text{(3a)} \\
& \begin{bmatrix} Q & CR + DL \\ (CR + DL)^\top & R + R^\top - P \end{bmatrix} \succ 0 && \text{(3b)} \\
& \mathrm{Tr}(Q) \leq \gamma^2 && \text{(3c)} \\
& P \in \mathbb{S}^n,\ Q \in \mathbb{S}^q,\ R \in \mathbb{R}^{n \times n},\ L \in \mathbb{R}^{m \times n}. && \text{(3d)}
\end{aligned}$$

<u>If feasible, then R is nons</u>ingular with valid $K = LR^{-1}$

[4]Thm. 5: De Oliveira, Mauricio C., José C. Geromel, and Jacques Bernussou. "Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems." International journal of control 75.9 (2002): 666-679.

Structured Robust Design

We desire $K = LR^{-1} \in \mathcal{S}$, but this is generally nonconvex

$$\text{top block: } \begin{bmatrix} P - EE^\top & AR + BL \\ (AR + BL)^\top & R + R^\top - P \end{bmatrix} \succ 0 \tag{4}$$

Only R and L are sparsity constrained, P can be dense

Standard approach: $L \in \mathcal{S}$, R is *diagonal* $\implies K \in \mathcal{S}$

Structured Control with Reduced Conservatism

R diagonal is too strict, can be loosened [5]

Given basis $\{S_\ell\}_{\ell=1}^k$ for $\mathcal{S}$ with ($\mathcal{S} = \text{span}(\{S_\ell\})$)

Define *representation* S and convex set $\Upsilon(S)$ as

$$S := \begin{bmatrix} S_1 & S_2 & \dots & S_k \end{bmatrix} \tag{5}$$

$$\Upsilon(S) := \{Q \in \mathbb{R}^{n\times n} \mid \exists \Lambda \in \mathbb{S}^k : S(I_k \otimes Q) = S(\Lambda \otimes I_n)\} \tag{6}$$

Subspace-compatible controller

$$L \in \mathcal{S},\ R \in \Upsilon(S),\ |R| \neq 0 \implies LR^{-1} \in \mathcal{S} \tag{7}$$

[5] Ferrante, Francesco, Fabrizio Dabbene, and Chiara Ravazzi. "On the design of structured stabilizers for LTI systems." IEEE Control Systems Letters 4.2 (2019): 289-294.

Convex-Relaxed Structured Robust Design

H_2 norm $< \gamma$ for $u = Kx$ with $K \in \mathcal{S}$ if program is feasible

$$\begin{aligned}
\operatorname*{find}_{P,Q,R,L} \quad & \begin{bmatrix} P - EE^\top & AR + BL \\ (AR+BL)^\top & R + R^\top - P \end{bmatrix} \succ 0 && \text{(8a)} \\
& \begin{bmatrix} Q & CR + DL \\ (CR+DL)^\top & R + R^\top - P \end{bmatrix} \succ 0 && \text{(8b)} \\
& \operatorname{Tr}(Q) \leq \gamma^2 && \text{(8c)} \\
& P \in \mathbb{S}^n, \; Q \in \mathbb{S}^q, \; R \in \Upsilon(\mathcal{S}), \; L \in \mathcal{S}. && \text{(8d)}
\end{aligned}$$

If feasible, then R is nonsingular with valid $K = LR^{-1} \in \mathcal{S}$

Still requires *known* plant (A, B)

Structured Data-Driven Control

Algorithms for Direct Data-Driven Control

Virtual Reference Feedback Tuning (first methods)

Set-Membership (this talk)

- (Data-consistent plants) $\subseteq$ (K-Stabilized plants)
- Certificates: Farkas, Interval, **S-Lemma (QMI)**, SOS

Behavioral

- Description of all signal relations
- Parameterize and pick out best system trajectory (DeepC)

Others: Koopman, Gaussian Processes

Data Collected

Data (x, u) collected as a T-length trajectory:

$$\begin{aligned}
\mathbf{X}_- &:= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} && \in \mathbb{R}^{n \times T} \\
\mathbf{U} &:= \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix} && \in \mathbb{R}^{m \times T} \\
\mathbf{X}_+ &:= \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix} && \in \mathbb{R}^{n \times T}
\end{aligned}$$

Unknown (but bounded) observation noise process w

$$\mathbf{W} := \begin{bmatrix} w(0) & w(1) & \dots & w(T-1) \end{bmatrix} \in \mathbb{R}^{n \times T} \tag{9}$$

Ground truth (A_*, B_*) obeys

$$\mathbf{X}_+ = A_* \mathbf{X}_- + B_* \mathbf{U} + \mathbf{W} \tag{10}$$

Noise Description

Noise sequence w bounded inside matrix ellipsoid [6]:

$$\begin{bmatrix} I \\ \mathbf{W}^\top \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{W}^\top \end{bmatrix}^\top \succeq 0. \tag{11}$$

for $\Phi \in \mathbb{S}^{n+T}$ with $\Phi_{11} \in \mathbb{S}^n_+$ and $-\Phi_{22} \in \mathbb{S}^T_{++}$

Process noise bound $\forall t : \ \|w(t)\|_2 \leq \epsilon$ (overapproximation)

$$\Phi = \begin{bmatrix} T\epsilon I_n & \mathbf{0} \\ \mathbf{0} & -I_T \end{bmatrix}. \tag{12}$$

[6] Van Waarde, Henk J., et al. "Quadratic matrix inequalities with applications to data-based control." SIAM Journal on Control and Optimization 61.4 (2023): 2251-2281.

Data-Consistency Set

Data-bound-characterizing matrix Ψ

$$\Psi := \begin{bmatrix} I & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U} \end{bmatrix}^\top \Phi \begin{bmatrix} I & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U} \end{bmatrix}. \tag{13}$$

Set of data-consistent plants (A, B):

$$\Sigma_{\mathcal{D}} = \left\{ (A, B) \;\middle|\; \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \Psi \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succeq 0 \right\}. \tag{14}$$

Control Consistency Set

Equivalent statements:

$$\begin{bmatrix} P - EE^\top & AR + BL \\ (AR + BL)^\top & R + R^\top - P \end{bmatrix} \succ 0$$

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \left[\begin{matrix} P - EE^\top & \mathbf{0} \\ \mathbf{0} & - \begin{bmatrix} R \\ L \end{bmatrix} (R + R^\top - P)^{-1} \begin{bmatrix} R \\ L \end{bmatrix}^\top \end{matrix} \right] \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succ 0$$

Must be enforced $\forall (A, B) \in \Sigma_{\mathcal{D}}$ with common (P, R, L)

Control over Common Set

Matrix S-Lemma[7]: exist constants $\alpha \geq 0$, $\beta > 0$ such that:

$$\begin{bmatrix} P - EE^\top - \beta I & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} R \\ L \end{bmatrix} (R + R^\top - P)^{-1} \begin{bmatrix} R \\ L \end{bmatrix}^\top \end{bmatrix} - \alpha \Psi \succeq 0.$$

Nonconservative (over 1 quadratic constraint)

[7]Van Waarde, Henk J., M. Kanat Camlibel, and Mehran Mesbahi. "From noisy data to feedback controllers: Nonconservative design via a matrix S-lemma." IEEE Transactions on Automatic Control 67.1 (2020): 162-175.

Control over Common Set

Expand by Schur Complement

$$\left[\begin{array}{ccc:c} P - EE^\top - \beta I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & L \\ \hdashline \mathbf{0} & R^\top & L^\top & R + R^\top - P \end{array}\right] - \alpha \left[\begin{array}{c:c} \Psi & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} \end{array}\right] \succ 0$$

Now is an LMI (convex expression) in (R, L, P)

Data-Driven Structured Control

$$\inf_{P,R,L,\alpha,\beta,\gamma} \quad \gamma \quad \text{subject to:}$$

$$\left[\begin{array}{ccc:c} P-EE^\top-\beta I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & L \\ \hdashline \mathbf{0} & R^\top & L^\top & R+R^\top-P \end{array}\right] - \alpha \left[\begin{array}{c:c} \Psi & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} \end{array}\right] \succeq 0 \tag{15a}$$

$$\begin{bmatrix} Q & CR+DL \\ (CR+DL)^\top & R+R^\top-P \end{bmatrix} \succeq 0 \tag{15b}$$

$$\mathrm{Tr}(Q) \leq \gamma^2 \tag{15c}$$

$$\alpha \geq 0, \beta > 0, \gamma \geq 0 \tag{15d}$$

$$P \in \mathbb{S}^n,\ Q \in \mathbb{S}^q,\ R \in \Upsilon(S),\ L \in \mathcal{S} \tag{15e}$$

Examples

Three-State Two-Input System

Ground truth of

$$A_* = \begin{bmatrix} -0.4095 & 0.4036 & -0.0874 \\ 0.5154 & -0.0815 & 0.1069 \\ 1.6715 & 0.7718 & -0.3376 \end{bmatrix} \tag{16}$$

$$B_* = \begin{bmatrix} 0 & 0 \\ -0.6359 & -0.1098 \\ -0.0325 & 2.2795 \end{bmatrix}$$

H_2-suboptimal control with

$$C = \begin{bmatrix} I_3 \\ \mathbf{0}_{2\times 3} \end{bmatrix}, \qquad D = \begin{bmatrix} \mathbf{0}_{3\times 2} \\ I_2 \end{bmatrix}, \qquad E = I_3. \tag{17}$$

Sparse Control Task

Controller structure:

$$K \in \begin{bmatrix} \bullet & \bullet & 0 \\ 0 & \bullet & \bullet \end{bmatrix} \qquad \Upsilon(S) : \begin{bmatrix} R_{11} & R_{12} & 0 \\ 0 & R_{22} & 0 \\ 0 & R_{32} & R_{33} \end{bmatrix} \tag{18}$$

Control tasks:

1. Unstructured (dense, comparison)
2. $P = R$ diagonal, $L \in \mathcal{S}$
3. P dense, R diagonal, $L \in \mathcal{S}$
4. P dense, $R \in \Upsilon(S)$, $L \in \mathcal{S}$ (our scheme)

Comparison (increased noise bound)

Closed-loop H_2 upper-bounds v.s. ϵ with $T = 20$

Design	(A_*, B_*)	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.15$
1 (Unstructured)	2.1537	2.3448	3.0939	4.5757
2 ($P = R$ diag.)	3.5658	4.6619	7.4193	Infeasible
3 (R diag.)	3.0089	3.5997	4.9506	9.1999
4 ($R \in \Upsilon(S)$)	2.9794	3.5495	4.6806	8.9710

Comparison (increased time horizon)

Closed-loop H_2 upper-bounds v.s. T with $\epsilon = 0.1$

Design	(A_*, B_*)	$T = 6$	$T = 10$	$T = 20$
1 (Unstructured)	2.1537	2.9911	2.8156	3.0939
2 ($P = R$ diag.)	3.5658	6.3386	7.0963	7.4193
3 (R diag.)	3.0089	4.5545	4.5044	4.9506
4 ($R \in \Upsilon(S)$)	2.9794	4.4036	4.4323	4.6806

Non-monotonicity due to overapproximation

Take-aways

Conclusion

Structured H_2 regulation based on data

Convex $\Upsilon(S)$ description of $\mathcal{S}$ constraint

Matrix S-Lemma for robust certification

Extensions: LPV synthesis, determining optimal structures

Thanks!