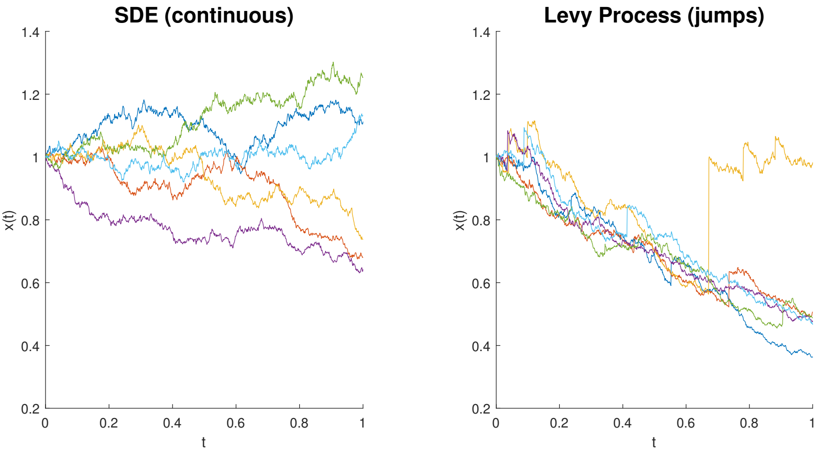
Risk Analysis for Stochastic Processes using Polynomial Optimization

Jared Miller
Matteo Tacchi
Mario Sznaier
Ashkan Jasour

MA32: INFORMS Annual Meeting

Stochastic Process

Nonanticipative, indep.-increment set of prob. dists. $\{\mu_t\}$



Geometric Brownian motion (left), Merton jump diffusion (right)

Questions to ask

Given a state function $p(x)$ (e.g. height, voltage):

What is the maximum along stochastic trajectories of the:

- Mean of p?
- Quantile statistics of p?
- Conditional Value-at-Risk of p?

Risk, safety analysis.

Examples of Generators

Generator $\mathcal{L}$ of process $\forall v \in \mathrm{dom}(\mathcal{L}) = \mathcal{C}$:

$$\mathcal{L}_\tau v = \lim_{\tau' \to \tau} \left(\mathbb{E}[v(t+\tau', x) \mid \mu_{t+\tau'}] - v(t,x)\right)/\tau' \tag{1}$$

Discrete-time Markov Process

$$X_{t+\tau} = F(t, X_t, \omega_t), \qquad \omega_t \sim \xi \text{ (sampled)} \tag{2}$$

$$\mathcal{L}_\tau v = \left(\int_\Omega v\left(t+\tau, F(t,x,\omega)\right) d\xi(\omega) - v\right)/\tau. \tag{3}$$

Stochastic Differential Equation

$$dx = f(t,x)dt + g(t,x)dW, \tag{4}$$

$$\mathcal{L}_0 v = \partial_t v + f \cdot \nabla_x v + g^T(\nabla^2_{xx} v)g/2 \tag{5}$$

Others: Lévy processes, hybrid, switching, time-delay

Chance-Peak Problem

Distribution $p_{\#}\mu_t$ of $p(x(t))$

What is the maximum risk R along the stochastic trajectory?

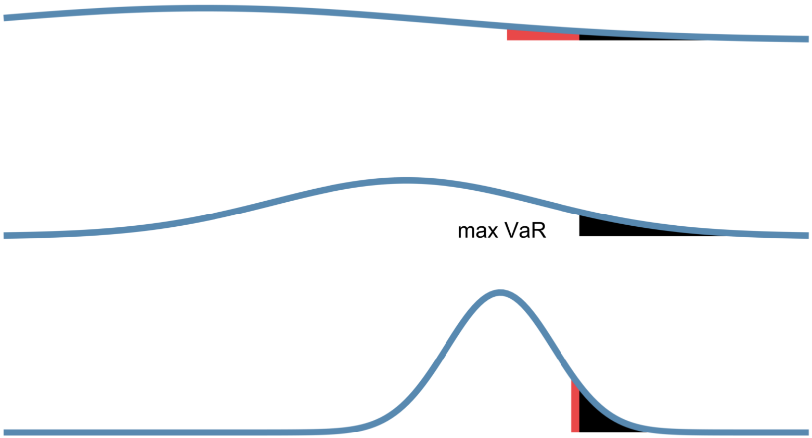
$$P^* = \sup_{t^* \in [0, T]} \quad R(p_{\#}\mu_{t^*}) \tag{6a}$$

$$x(t) \text{ follows } \mathcal{L} \qquad \forall t \in [0, t^*] \tag{6b}$$

$$x(0) \sim \mu_0 \tag{6c}$$

Maximal Value at Risk

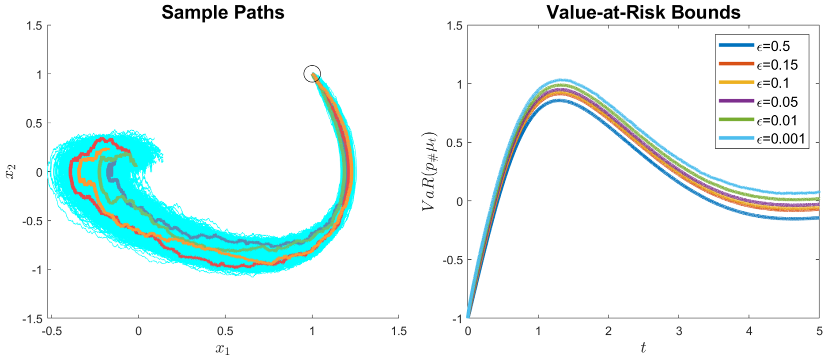
Maximize ϵ-VaR among multiple distributions



Red + Black areas = 10% probability

Value-at-Risk Example (Monte Carlo)

50,000 samples with $T = 5,\ \Delta t = 10^{-3}$



$$\text{VaR of } p = -x_2 \text{ along } dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

Chance-Peak Measure Programs

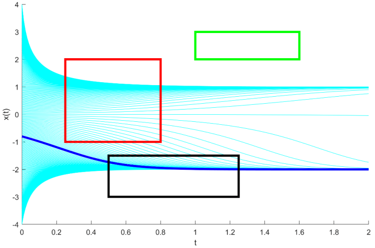
Occupation measures

Avg. time trajectories spend in set

Test function $v(t, x) \in \mathcal{C}$

Single trajectory: $\langle v, \mu \rangle = \mathbb{E}_{x(0)=x_0}\left[\int_0^T v(t, x(t \mid x_0))dt\right]$

Averaged trajectory: $\langle v, \mu \rangle = \mathbb{E}_{x(0)\sim\mu_0}\left[\int_0^T v(t, x(t \mid x_0))dt\right]$



$$x' = -x(x+2)(x-1)$$

Martingale Relation

$$\forall v : \ \mathbb{E}[v(t+s,x) \mid X_{t+s}] = \mathbb{E}[v(t,x) \mid X_t] + \int_{s'=t}^{t+s} \mathbb{E}[\mathcal{L}_0 v(t,x) \mid X_{s'}] \quad (7)$$

Relation between measures (μ_t, μ_{t+s}, μ)

$$\langle v(t+s,x), \mu_{t+s} \rangle = \langle v(t,x), \mu_t \rangle + \langle \mathcal{L}_0 v(t',x), \mu \rangle \quad (8)$$

Shorthand notation (adjoint)

$$\mu_{t+s} = \mu_t + \mathcal{L}_0^\dagger \mu \quad (9)$$

Triple of (8) is supported on graph of $\mathcal{L}_0$ (assuming compact + regularity), similar for [sum of $\mathcal{L}_\tau$].

Mean Maximization

Infinite-dimensional Linear Program (Cho, Stockbridge, 2002)

$$p^* = \sup \quad \langle p(x), \mu_p \rangle \tag{10a}$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{10b}$$

$$\mu, \mu_p \in \mathcal{M}_+([0, T] \times X) \tag{10c}$$

Instance of Optimal Control Program (Lewis and Vinter, 1980)

(μ_p^*, μ^*) is feasible with $P^* = \langle p(x), \mu_p^* \rangle \leq p^*$

$P^* = p^*$ if compactness, regularity properties hold

Value-at-Risk Bounds

VaR is nonconvex, nonsubadditive

Concentration inequalities can upper-bound VaR

$$VaR_\epsilon(\xi) \leq \mathrm{stdev}(\xi) r + \mathrm{mean}(\xi)$$

Name	r	**Valid Condition**
Cantelli	$\sqrt{1/(\epsilon) - 1}$	ξ probability distribution
VP	$\sqrt{4/(9\epsilon) - 1}$	ξ unimodal, $\epsilon < 1/6$

Conditional Value at Risk (CVaR) can also bound VaR

Concentration-Bounded Chance-Peak

Apply concentration inequalities to get upper bound $P_r^* \geq P^*$

Objective upper-bounds VaR w.r.t. time-t^* distribution μ_{t^*}

$$
\begin{aligned}
P_r^* = \sup_{t^* \in [0,T]} \quad & r\sqrt{\langle p^2, \mu_{t^*}\rangle - \langle p, \mu_{t^*}\rangle^2} + \langle p, \mu_{t^*}\rangle && \text{(11a)} \\
& x \text{ follows } \mathcal{L} && \text{(11b)} \\
& x(0) \sim \mu_0 && \text{(11c)}
\end{aligned}
$$

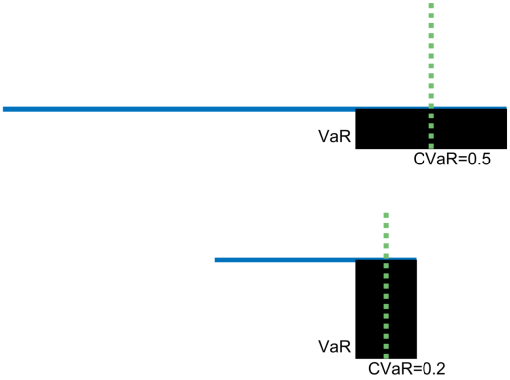
SOCP in measures (3d SOC) for $p_r^* \geq P_r^*$

Same constraints as mean-maximization, different objective

Conditional Value-at-Risk

CVAR: Average quantity above the Value-at-Risk

$$CVaR_\epsilon(\xi(\omega)) = (1/\epsilon) \int_{\omega \geq VaR_\epsilon(\xi)} \omega d\xi(\omega)$$



Uniform distributions with same VaR, different CVaR (70%)

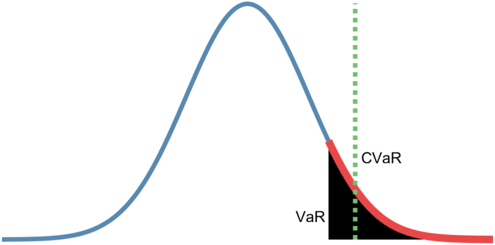
CVaR Linear Program

Measure LP to compute CVaR (with $\frac{d\psi}{d\xi} \leq \frac{1}{\epsilon}$)

$$CVaR_\epsilon(\nu) = \sup_{\psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R})} \langle \omega, \psi \rangle \tag{12a}$$

$$\epsilon\psi + \hat{\psi} = \nu \tag{12b}$$

$$\langle 1, \psi \rangle = 1 \tag{12c}$$



VaR $= 1.2816$, CVaR$= 1.7550$, $\epsilon\psi \leq \xi$

CVaR Chance-Peak

Highest CVaR along SDE trajectories

$$P_c^* = \sup_{t^* \in [0,T]} CVaR_\epsilon(p_\# \mu_{t^*}) \tag{13a}$$

$$x \text{ follows } \mathcal{L} \tag{13b}$$

$$x(0) \sim \mu_0 \tag{13c}$$

Almost the same as VaR chance-peak, with $P_c^* \geq P^*$

CVaR Measure program

Add CVaR objective, constraints to chance-peak

$$
\begin{aligned}
p_c^* = \ \sup \quad & \langle \omega, \psi \rangle && \text{(14a)}\\
& \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu && \text{(14b)}\\
& \langle 1, \psi \rangle = 1 && \text{(14c)}\\
& \epsilon\psi + \hat{\psi} = p_\# \mu_\tau && \text{(14d)}\\
& \mu, \mu_\tau \in \mathcal{M}_+([0, T] \times X) && \text{(14e)}\\
& \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R}) && \text{(14f)}
\end{aligned}
$$

Upper-bound $p_c^* \geq P_c^* \geq P^*$, LP in measures

Comparison of bounds

$P_r^* = p_r^*$ and $P_c^* = p_c^*$ if

1. $\mathcal{L}$ has unique solutions (e.g. SDE: Lipchitz, Growth)
2. $[0, T] \times X$ compact
3. $p(x)$ is continuous

$P_{\text{Cantelli}}^* \geq P_c^*$ always, but $(P_c^*,\ P_{\text{VP}}^*)$ incomparable (so far)

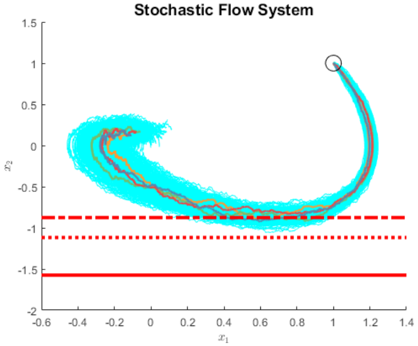
Empirically, degree-k moment LMIs satisfy $p_{\text{Cantelli},k}^* \geq p_{c,k}^*$

Chance-Peak Examples

Two-State

Stochastic Flow (Prajna, Rantzer) with $T = 5,\; p(x) = -x_2$

$$dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

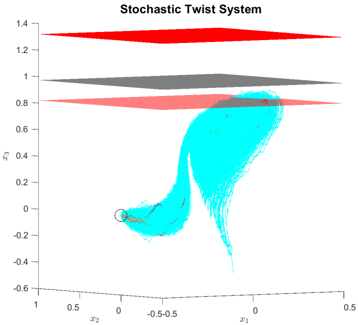


$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Three-State

Stochastic Twist system with $T = 5$, $p(x) = x_3$

$$dx = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} dw$$

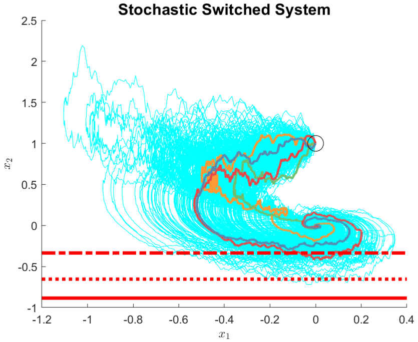


$d = 6$ (translucent=50%, gray=85% CVAR, solid=85% VP)

Two-State Switching

Switching subsystems at $T = 5$, $p(x) = -x_2$

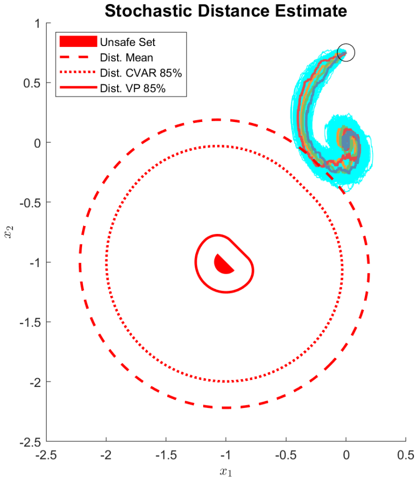
$$dx = \left\{ \begin{bmatrix} -2.5x_1 - 2x_2 \\ -0.5x_1 - x_2 \end{bmatrix}, \begin{bmatrix} -x_1 - 2x_2 \\ 2.5x_1 - x_2 \end{bmatrix} \right\} dt + \begin{bmatrix} 0 \\ 0.25x_2 \end{bmatrix} dw$$



$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Two-State Distance

Maximize VaR of (negative) L_2 distance to X_u



$d = 6$ (dash-dot=50%, dotted=85% CVaR, solid=85% VP)

Take-aways

Conclusion

Posed the chance-peak problem for wide class of $\mathcal{L}$

Solved using infinite-dimensional SOCPs, LPs in measures

Certified outer-approximations of risk

Acknowledgements

Roy Smith, Automatic Control Lab (IfA)

POP group at LAAS-CNRS

NCCR Automation

Air Force Office for Scientific Research

National Science Foundation

Analyze risk using polynomials

arxiv:2303.16064

`http://github.com/jarmill/chance_peak`

`http://github.com/jarmill/cvar_peak`

SOS Expectation-Peak

$$
\begin{aligned}
d^*_{\mathbb{E}} = \min \quad & \int_X v(0,x)\; d\mu_0(x) && (15a)\\
& -\mathcal{L}v(t,x) \in \Sigma[[0,T]\times X] && (15b)\\
& v(t,x) - p(x) \in \Sigma[[0,T]\times X] && (15c)\\
& v \in \mathbb{R}[t,x]. && (15d)
\end{aligned}
$$

SOS Concentration-Peak

$$
\begin{aligned}
d_r^* = \inf \quad & u_1 + 2u_3 + \int_{X_0} v(0, x_0) d\mu_0(x_0) && \text{(16a)}\\
& -\mathcal{L}v(t,x) \in \Sigma[[0,T] \times X] && \text{(16b)}\\
& v(t,x) + u_1\, p^2(x) - 2\, u_2\, p(x) - p(x) && \text{(16c)}\\
& \qquad \in \Sigma[[0,T] \times X] && \\
& ([u_1 + u_3, -(r/2), u_2], u_3) \in \mathbb{L}^3 && \text{(16d)}\\
& u \in \mathbb{R}^3, \ v \in \mathcal{C}([0,T] \times X). &&
\end{aligned}
$$

SOS CVaR-Peak

$$
\begin{aligned}
d_c^* = \min \quad & u + \int_X v(0,x)\ d\mu_0(x) && \text{(17a)}\\
& -\mathcal{L}v(t,x) \in \Sigma[[0,T] \times X] && \text{(17b)}\\
& v(t,x) - w(p(x)) \in \Sigma[[0,T] \times X] && \text{(17c)}\\
& u + \epsilon\, w(q) - q \in \Sigma[p_{min}, p_{max}] && \text{(17d)}\\
& w(q) \in \Sigma[p_{min}, p_{max}] && \text{(17e)}\\
& u \in \mathbb{R}, v \in \mathbb{R}[t,x]. && \text{(17f)}
\end{aligned}
$$