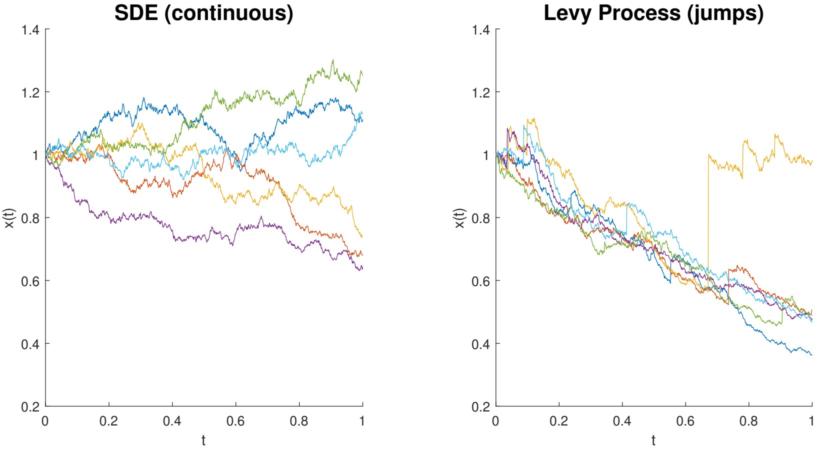
Peak Value-At-Risk Estimation for Stochastic Differential Equations Using Occupation Measures

Jared Miller
Matteo Tacchi
Mario Sznaier
Ashkan Jasour

ThB22.6: IEEE CDC

Stochastic Process

Nonanticipative, indep.-increment set of prob. dists. $\{\mu_t\}$



Geometric Brownian motion (left), Merton jump diffusion (right)

Questions to ask

Given a state function $p(x)$ (e.g. height, voltage):

What is the maximum along stochastic trajectories of the:

- Mean of p?
- Quantile statistics of p?
- Conditional Value-at-Risk of p?

Risk, safety analysis.

Examples of Generators

Generator $\mathcal{L}$ of process $\forall v \in \mathrm{dom}(\mathcal{L}) = \mathcal{C}$:

$$\mathcal{L}_\tau v = \lim_{\tau' \to \tau} \left(\mathbb{E}[v(t+\tau', x) \mid \mu_{t+\tau'}] - v(t,x) \right) / \tau' \tag{1}$$

Discrete-time Markov Process ($\mathcal{C} = C([0, T] \times X)$)

$$X_{t+\tau} = F(t, X_t, \omega_t), \qquad \omega_t \sim \xi \text{ (sampled)} \tag{2}$$

$$\mathcal{L}_\tau v = \left(\int_\Omega v\left(t+\tau, F(t,x,\omega)\right) d\xi(\omega) - v \right) / \tau \tag{3}$$

Examples of Generators (Continuous-time)

Generator $\mathcal{L}$ of process $\forall v \in \mathrm{dom}(\mathcal{L}) = \mathcal{C}$:

$$\mathcal{L}_\tau v = \lim_{\tau' \to \tau} \left(\mathbb{E}[v(t+\tau', x) \mid \mu_{t+\tau'}] - v(t,x) \right) / \tau' \tag{4}$$

Stochastic Differential Equation ($\mathcal{C} = C^{1,2}([0, T] \times X)$)

$$dx = f(t,x)dt + g(t,x)dW, \tag{5}$$

$$\mathcal{L}_0 v = \partial_t v + f \cdot \nabla_x v + g^T (\nabla^2_{xx} v) g/2 \tag{6}$$

Others: Lévy processes, hybrid, switching, time-delay

Chance-Peak Problem

Distribution $p_\# \mu_t$ of $p(x(t))$ (pushforward)

What is the maximum risk R along the stochastic trajectory?

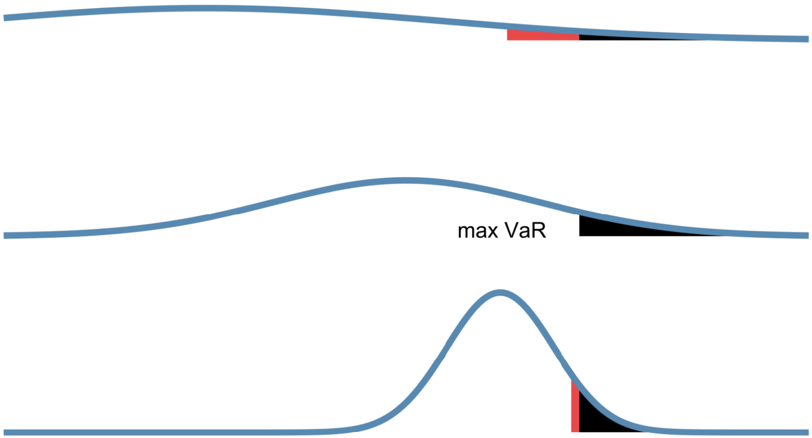
$$P^* = \sup_{t^* \in [0, T]} \quad R(p_\# \mu_{t^*}) \tag{7a}$$

$$x(t) \text{ follows } \mathcal{L} \qquad \forall t \in [0, t^*] \tag{7b}$$

$$x(0) \sim \mu_0 \tag{7c}$$

Maximal Value at Risk

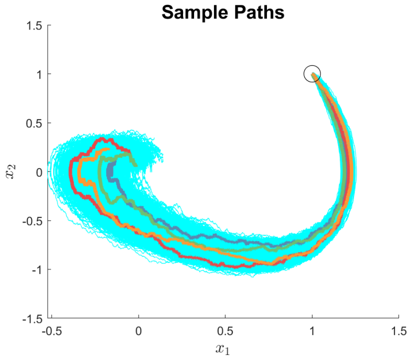
Maximize ϵ-VaR among multiple distributions

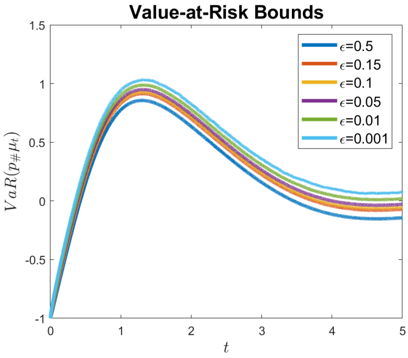


Red + Black areas = 10% probability

Value-at-Risk Example (Monte Carlo)

50,000 samples with $T = 5$, $\Delta t = 10^{-3}$





$$\text{VaR of } p = -x_2 \text{ along } dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

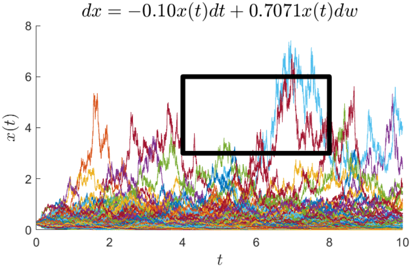
Chance-Peak Measure Programs

Occupation measures

Avg. time trajectories spend in set

Test function $v(t, x) \in \mathcal{C}$

Averaged value:
$\langle v, \mu \rangle = \int_0^T \mathbb{E}_{x \sim X_t}[v(t, x)]dt$



Martingale Relation

$$\forall v: \ \mathbb{E}[v(t+s,x) \mid X_{t+s}] = \mathbb{E}[v(t,x) \mid X_t] + \int_{s'=t}^{t+s} \mathbb{E}[\mathcal{L}_0 v(t,x) \mid X_{s'}] \tag{8}$$

Relation between measures (μ_t, μ_{t+s}, μ)

$$\langle v(t+s,x), \mu_{t+s}\rangle = \langle v(t,x), \mu_t\rangle + \langle \mathcal{L}_0 v(t',x), \mu\rangle \tag{9}$$

Shorthand notation (adjoint)

$$\mu_{t+s} = \mu_t + \mathcal{L}_0^\dagger \mu \tag{10}$$

Triple of (9) is supported on graph of $\mathcal{L}_0$ (assuming compact + regularity), similar for [sum of $\mathcal{L}_\tau$].

Mean Maximization

Infinite-dimensional Linear Program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \ \sup \quad & \langle p, \mu_p \rangle && \text{(11a)} \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu && \text{(11b)} \\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) && \text{(11c)}
\end{aligned}
$$

Instance of Optimal Control Program (Lewis and Vinter, 1980)

(μ_p^*, μ^*) is feasible with $P^* = \langle p(x), \mu_p^* \rangle \leq p^*$

$P^* = p^*$ if compactness, regularity properties hold

Value-at-Risk Bounds

VaR is nonconvex, nonsubadditive

Concentration inequalities can upper-bound VaR

$$VaR_\epsilon(\xi) \leq \mathrm{stdev}(\xi) r + \mathrm{mean}(\xi)$$

Name	r	Valid Condition
Cantelli	$\sqrt{1/(\epsilon) - 1}$	ξ probability distribution
VP	$\sqrt{4/(9\epsilon) - 1}$	ξ unimodal, $\epsilon < 1/6$

Conditional Value at Risk (CVaR) can also bound VaR

Concentration-Bounded Chance-Peak

Apply concentration inequalities to get upper bound $P_r^* \geq P^*$

Objective upper-bounds VaR w.r.t. time-t^* distribution μ_{t^*}

$$P_r^* = \sup_{t^* \in [0,T]} r\sqrt{\langle p^2, \mu_{t^*}\rangle - \langle p, \mu_{t^*}\rangle^2} + \langle p, \mu_{t^*}\rangle \tag{12a}$$

$$x \text{ follows } \mathcal{L} \tag{12b}$$

$$x(0) \sim \mu_0 \tag{12c}$$

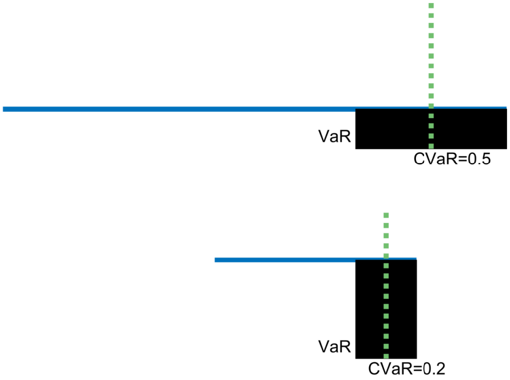
SOCP in measures (3d SOC) for $p_r^* \geq P_r^*$

Same constraints as mean-maximization, different objective

Conditional Value-at-Risk

CVAR: Average quantity above the Value-at-Risk

$$CVaR_\epsilon(\xi(\omega)) = (1/\epsilon)\int_{\omega \geq VaR_\epsilon(\xi)} \omega d\xi(\omega)$$



Uniform distributions with same VaR, different CVaR (70%)

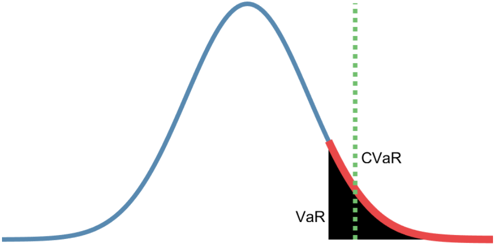
CVaR Linear Program

Measure LP to compute CVaR (with $\frac{d\psi}{d\xi} \leq \frac{1}{\epsilon}$)

$$CVaR_\epsilon(\nu) = \sup_{\psi,\hat{\psi}\in\mathcal{M}_+(\mathbb{R})} \mathrm{mean}(psi) \tag{13a}$$

$$\epsilon\psi + \hat{\psi} = \nu \tag{13b}$$

$$\langle 1, \psi \rangle = 1 \tag{13c}$$



VaR $= 1.2816$, CVaR$= 1.7550$, $\epsilon\psi \leq \xi$

CVaR Chance-Peak

Highest CVaR along SDE trajectories

$$P_c^* = \sup_{t^* \in [0, T]} CVaR_\epsilon(p_\# \mu_{t^*}) \tag{14a}$$

$$x \text{ follows } \mathcal{L} \tag{14b}$$

$$x(0) \sim \mu_0 \tag{14c}$$

Almost the same as VaR chance-peak, with $P_c^* \geq P^*$

CVaR Measure program

Add CVaR objective, constraints to chance-peak

$$\begin{aligned}
p_c^* = \sup \quad & \mathrm{mean}(\psi) && (15a)\\
& \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu && (15b)\\
& \langle 1, \psi \rangle = 1 && (15c)\\
& \epsilon\psi + \hat{\psi} = p_\# \mu_\tau && (15d)\\
& \mu, \mu_\tau \in \mathcal{M}_+([0, T] \times X) && (15e)\\
& \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R}) && (15f)
\end{aligned}$$

Upper-bound $p_c^* \geq P_c^* \geq P^*$, LP in measures

Comparison of bounds

$P_r^* = p_r^*$ and $P_c^* = p_c^*$ if

1. $\mathcal{L}$ has unique solutions (e.g. SDE: Lipchitz, Growth)
2. $[0, T] \times X$ compact
3. $p(x)$ is continuous

$P_{\text{Cantelli}}^* \geq P_c^*$ always, but $(P_c^*,\ P_{\text{VP}}^*)$ incomparable (so far)

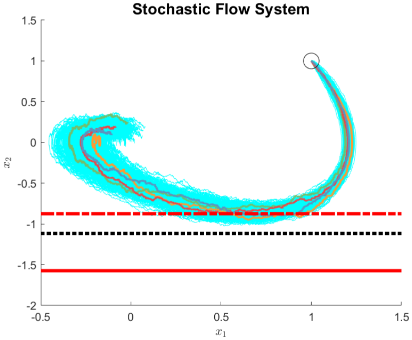
Empirically, degree-k moment LMIs satisfy $p_{\text{Cantelli},k}^* \geq p_{c,k}^*$

Chance-Peak Examples

Two-State

Stochastic Flow (Prajna, Rantzer) with $T = 5,\ p(x) = -x_2$

$$dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

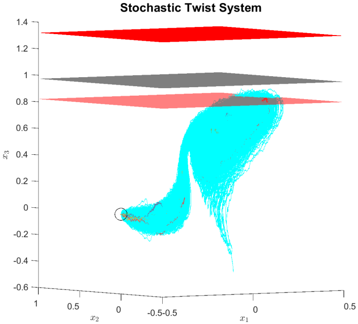


$d = 6$ (dash-dot=50%, dotted-black=85% CVAR, solid=85% VP)

Three-State

Stochastic Twist system with $T = 5$, $p(x) = x_3$

$$dx = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} dw$$

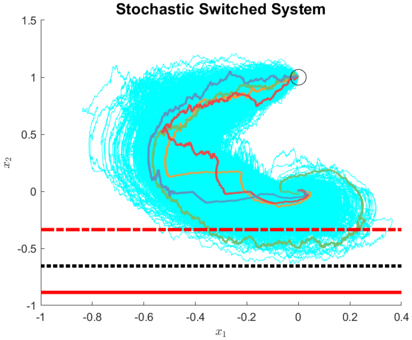


$d = 6$ (translucent=50%, gray=85% CVAR, solid=85% VP)

Two-State Switching

Switching subsystems at $T = 5$, $p(x) = -x_2$

$$dx = \left\{ \begin{bmatrix} -2.5x_1 - 2x_2 \\ -0.5x_1 - x_2 \end{bmatrix}, \begin{bmatrix} -x_1 - 2x_2 \\ 2.5x_1 - x_2 \end{bmatrix} \right\} dt + \begin{bmatrix} 0 \\ 0.25x_2 \end{bmatrix} dw$$

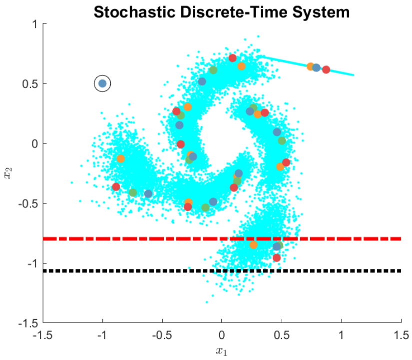


$d = 6$ (dash-dot=50%, dotted-black=85% CVAR, solid=85% VP)

Two-State Discrete-Time

Parameter λ sampled from $\lambda[t] \in \mathcal{N}(0,1)$

$$x_+ = \begin{bmatrix} -0.3x_1 + 0.8x_2 + x_1 x_2 \lambda/4 \\ -0.9x_1 - -0.1x_2 - 0.2x_1^2 + \lambda/40 \end{bmatrix}.$$



$d = 6$ (dash-dot=50%, black-dotted=85% CVaR)

Take-aways

Conclusion

Posed the chance-peak problem for wide class of $\mathcal{L}$

Solved using infinite-dimensional SOCPs, LPs in measures

Certified outer-approximations of risk

Acknowledgements

Roy Smith, Automatic Control Lab (IfA)

POP group at LAAS-CNRS

NCCR Automation

Air Force Office for Scientific Research

National Science Foundation

Thanks!

Questions?

SOS Expectation-Peak

$$
\begin{aligned}
d^*_{\mathbb{E}} = \min \quad & \int_X v(0,x)\ d\mu_0(x) && \text{(16a)} \\
& -\mathcal{L}v(t,x) \in \Sigma[[0,T] \times X] && \text{(16b)} \\
& v(t,x) - p(x) \in \Sigma[[0,T] \times X] && \text{(16c)} \\
& v \in \mathbb{R}[t,x]. && \text{(16d)}
\end{aligned}
$$

SOS Concentration-Peak

$$
\begin{aligned}
d_r^* = \inf \quad & u_1 + 2u_3 + \int_{X_0} v(0, x_0) d\mu_0(x_0) && \text{(17a)} \\
& -\mathcal{L}v(t, x) \in \Sigma[[0, T] \times X] && \text{(17b)} \\
& v(t, x) + u_1\, p^2(x) - 2\, u_2\, p(x) - p(x) && \text{(17c)} \\
& \qquad \qquad \in \Sigma[[0, T] \times X] && \\
& ([u_1 + u_3, -(r/2), u_2], u_3) \in \mathbb{L}^3 && \text{(17d)} \\
& u \in \mathbb{R}^3, \;\; v \in \mathcal{C}([0, T] \times X). &&
\end{aligned}
$$

SOS CVaR-Peak

$$
\begin{aligned}
d_c^* = \min \quad & u + \int_X v(0, x) \; d\mu_0(x) && \text{(18a)}\\
& -\mathcal{L}v(t, x) \in \Sigma[[0, T] \times X] && \text{(18b)}\\
& v(t, x) - w(p(x)) \in \Sigma[[0, T] \times X] && \text{(18c)}\\
& u + \epsilon\, w(q) - q \in \Sigma[p_{min}, p_{max}] && \text{(18d)}\\
& w(q) \in \Sigma[p_{min}, p_{max}] && \text{(18e)}\\
& u \in \mathbb{R}, v \in \mathbb{R}[t, x]. && \text{(18f)}
\end{aligned}
$$