Peak Estimation for Uncertain and Switched Systems

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Main Ideas

Occupation measures can be used for peak estimation

Formulate peak estimation problems with uncertain dynamics

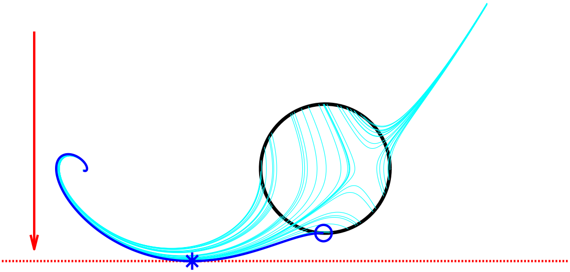
Find convergent approximations to peak values

Peak Estimation Problem

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3], \quad T = 5$$

Examples of Peak Estimation

Maximum infection rate in an epidemic

Maximum altitude of an aircraft

Maximum voltage on a powerline

Maximum speed of a motor

Maximum height of a shock wave

Maximum concentration of a chemical reagent

Peak and Measure Background

Measures

Nonnegative Borel Measure $\mu : \mathrm{Set}(X) \to \mathbb{R}_+$

$\mu \in \mathcal{M}_+(X)$: space of measures on X

$f \in C(X)$: continuous function on X

Duality pairing by Lebesgue integration:

$$\langle f(x), \mu \rangle = \int_X f(x) d\mu(x)$$

$\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

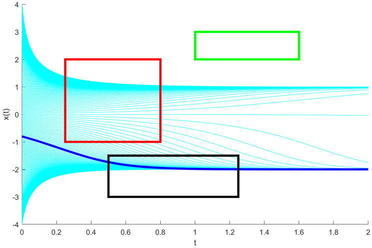
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

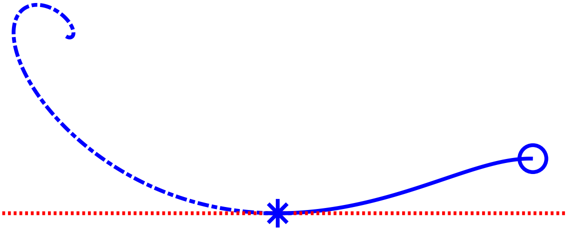
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x + 2)(x - 1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\mu_0^* : \qquad \langle v(0,x), \mu_0^* \rangle = v(0, x_0^*)$$

$$\mu_p^* : \qquad \langle v(t,x), \mu_p^* \rangle = v(t_p^*, x_p^*)$$

$$\mu^* : \qquad \langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p \rangle && \text{(1a)} \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1b)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(1c)} \\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) && \text{(1d)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0, T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Approximating Peak Estimation

Truncate infinite-dimensional LP

- Discretized (gridded) finite LPs
- Markov Chain Martingale
- **Moment-Sum-of-Squares Hierarchy**

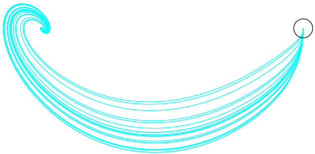
Moment-SOS: LMIs in increasing degree d,

$$p_d^* \geq p_{d+1}^* \geq \ldots p^*, \qquad \lim_{d \to \infty} p_d^* = p^*$$

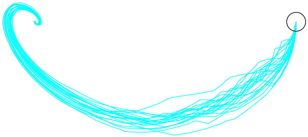
Uncertainty

System with Uncertainty Example

Time-Independent Uncertainty



Time-Dependent Uncertainty



$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1\theta - x_2 + x_1^3/3 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 w(t) - x_2 + x_1^3/3 \end{bmatrix}$$

$$\theta,\ w(t) \in [0.5, 1.5],\ x_0 = [1; 0]$$

Peak Estimation with Uncertainty

Time-independent $\theta \in \Theta$

Time-dependent $w(t) \in W, \ \forall t \in [0, T]$

$$P^* = \max_{t \in [0,T],\, x_0 \in X_0,\, \theta \in \Theta,\, w(t)} p(x(t \mid x_0, \theta, w(t)))$$

$$\dot{x}(t) = f(t, x(t), \theta, w(t)), \quad w(t) \in W \quad \forall t \in [0, T].$$

Uncertain Peak Measure Program

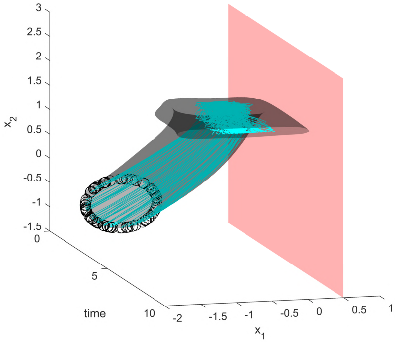
Based on optimal control LP of Lewis, Vinter (1980)

$$
\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p \rangle \\
& \langle v(t,x,\theta), \mu_p \rangle = \langle v(0,x,\theta), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x,\theta), \mu \rangle \quad \forall v \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu \in \mathcal{M}_+([0,T] \times X \times \Theta \times W) \\
& \mu_p \in \mathcal{M}_+([0,T] \times X \times \Theta) \\
& \mu_0 \in \mathcal{M}_+(X_0 \times \Theta)
\end{aligned}
$$

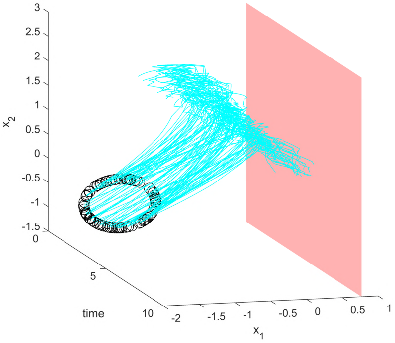
$P^* = p^*$ when f Lipschitz, $[0, T] \times X \times \Theta \times W$ compact

Continuous-Time Example

Maximize x_1 at order 4 with $w(t) \in [-0.2, 0.2]$



$\theta = 0$



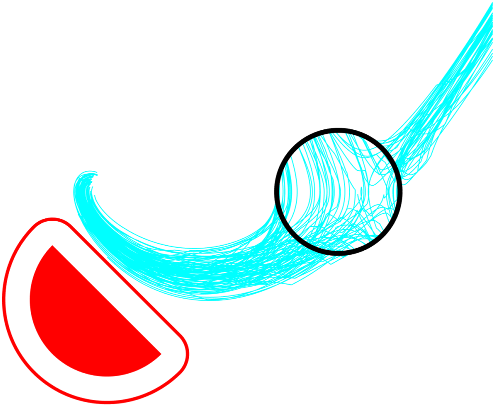
$\theta \in [-0.5, 0.5]$

$$\dot{x}(t) = \begin{bmatrix} -0.5x_1 - (0.5 + w(t))x_2 + 0.5 \\ -0.5x_2 + 1 + \theta \end{bmatrix}$$

Distance Uncertainty

Time dependent uncertainty $w(t) \in W \; \forall t \in [0, T]$

Uncertainty changes Liouville, Distance changes cost



L_2 bound of 0.1691

Switching

Switching Dynamics

Closed cover $\cup_{k=1}^{N_s} X^k = X$

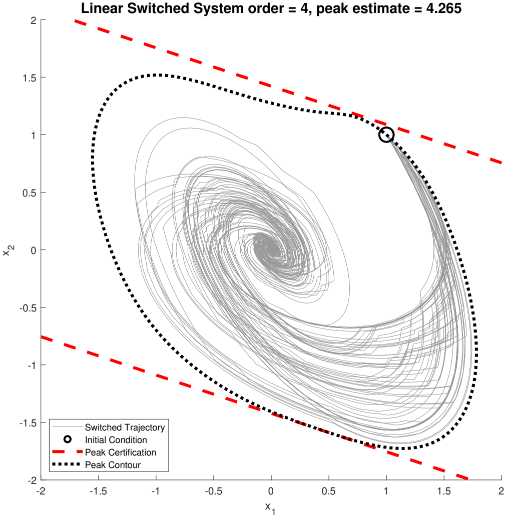
Dynamics $\dot{x} = f_k(t, x)$ admissible in set X^k

Per-subsystem occupation measures $\mu \in \mathcal{M}_+([0, T] \times X_k)$,

$$\mu = \sum_k \mu_k$$
$$\langle \mathcal{L}_f v(t, x), \mu \rangle = \sum_k \langle \mathcal{L}_{f_k} v(t, x), \mu_k \rangle$$

Switched LTI Impulse Response

Impulse response, switching between 2 linear systems



Discrete-Time Dynamics

Discrete-Time System

Transition dynamics $x_{t+1} = f(x_t), \quad \forall t = 0, \ldots, T-1$

Liouville equation

$$\langle v(x), \mu_p \rangle = \langle v(x), \mu_0 \rangle + \langle v(f(x)) - v(x), \mu \rangle \qquad \forall v$$

Liouville with uncertainty $x_{t+1} = f(x_t, \theta, w_t)$

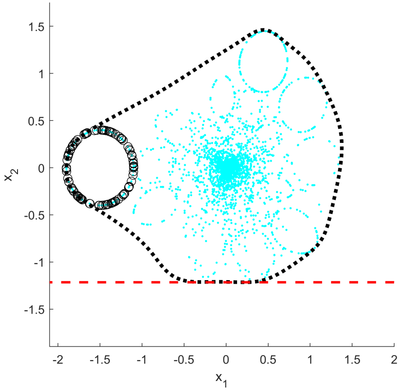
$$\langle v(x, \theta), \mu_p \rangle = \langle v(x, \theta), \mu_0 \rangle + \langle v(f(x, \theta, w)) - v(x), \mu \rangle \quad \forall v$$

Finite terminal time constraint

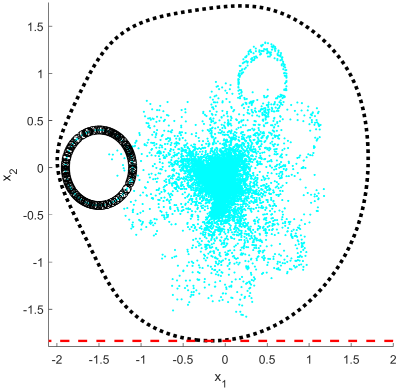
$$T \geq \langle 1, \mu \rangle$$

Discrete-Time Example

Discrete dynamics with switching and time-dependent uncertainty



$w_t = 0$



$w_t \in [-0.2, 0.2]$

Conclusion

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Extended peak estimation to problems with uncertainty

Time-independent, time-dependent, and switching processes

Continuous-time and discrete-time dynamics

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The Audience

Code: `http://github.com/jarmill/peak`