Peak Estimation Recovery and Safety Analysis

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Examples of Peak Estimation

Maximum infection rate in an epidemic

Maximum altitude of an aircraft

Maximum voltage on a powerline

Maximum speed of a motor

Maximum height of a shock wave

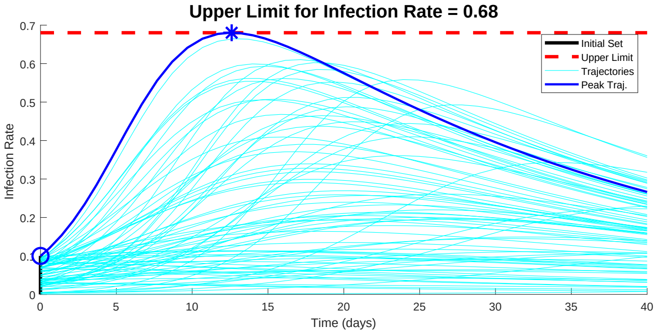
Maximum concentration of a chemical reagent

Peak Estimation for Epidemics

Susceptible, Infected, Removed

$$S' = -0.4SI, \qquad I' = 0.4SI - 0.04I, \qquad R' = 0.04I$$

$$S + I + R = 1, \qquad R_0 = 10$$



With $I_0 = 10\%$, $I_{max} = 68\%$ in 12.6 days

Peak Estimation Recovery

Peak Estimation Recovery Problem

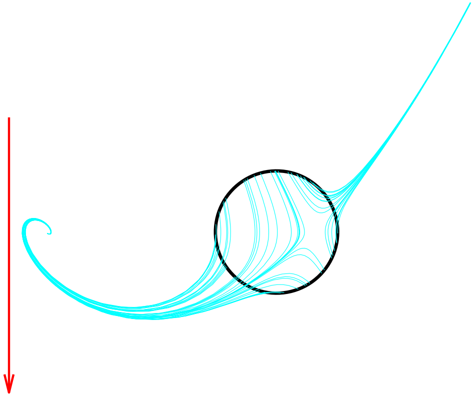
Estimation: Find peak value P^* of $p(x)$ in X:

$$\begin{aligned} P^* = \max_{t,\, x_0} \quad & p(x(t \mid x_0)) \\ & \dot{x}(t) = f(t, x) \quad t \in [0, T] \\ & x_0 \in X_0. \end{aligned}$$

Assume f locally Lipschitz

Recovery: If possible, find trajectory that achieves P^*

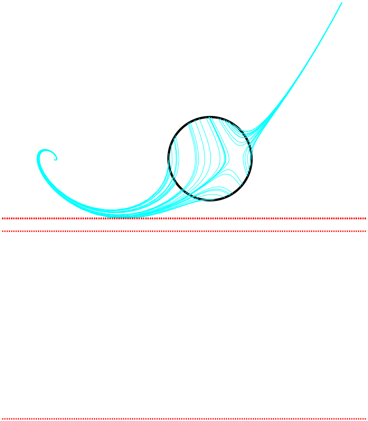
Peak Estimation Example System



Minimize x_2 on $\dot{x} = [x_2,\ -x_1 - x_2 + x_1^3/3]$

Initial set $X_0 = \{x \mid (x_1 - 1.5)^2 + x_2^2 \leq 0.4^2\}$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Optimal Trajectories

Optimal trajectories can be described by triples $(x_p^*,\ x_0^*,\ t_p^*)$

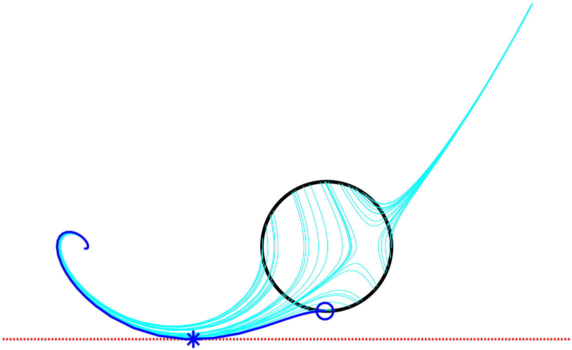
- x_p^*: location that attains p^*
- x_0^*: initial condition to produce x_p^*
- t_p^*: time to reach x_p^* from x_0^*

Triples satisfy relation:

$$P^* = p(x_p^*) = p(x(t_p^* \mid x_0^*))$$

Search for optimal points $(x_0, t_p, x_p) \in X_0 \times [0, T] \times X$

Peak Estimation Example Recovery



Optimal trajectory starts at $x_0^* = (1.4889, -0.3998)$

Attains peak at $x_p^* = (0.6767, -0.5734)$ in time $t_p^* = 1.6627$

Approximating Peak Estimation

Relax to infinite LP with bound $p^* \geq P^*$

LP Approximation methods:

- Discretize into finite LPs (Cho, Stockbridge, 2002)
- Sum-of-Squares Hierarchy (Fantuzzi, Goluskin, 2020)
- Moment Hierarchy (dual to sum-of-squares)

Measures and Functions

$\mu \in \mathcal{M}_+(X)$: Nonnegative signed borel measure on X

Is a function $\mu : \mathrm{Set}(X) \to \mathbb{R}_+$

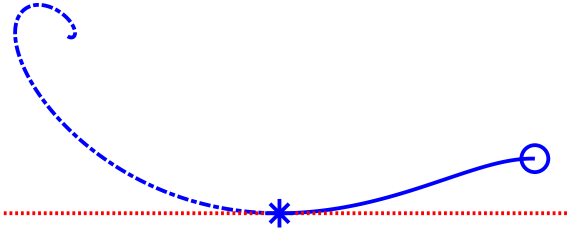
$f \in C(X)$: continuous function on X

Pairing $\langle f, \mu \rangle = \int_X f \; d\mu = \int_X f(x)\mu(x)dx$

Moments of μ : $y_\alpha = \langle x^\alpha, \mu \rangle$

Mass: $\mu(X) = y_0 = \langle 1, \mu \rangle$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\mu_0^*: \qquad \langle v(0, x), \mu_0 \rangle = v(0, x_0^*)$$

$$\mu_p^*: \qquad \langle v(t, x), \mu_p \rangle = v(t_p^*, x_p^*)$$

$$\mu^*: \qquad \langle v(t, x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \ &\max \quad \langle p(x), \mu_p \rangle && \text{(1a)}\\
&\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1b)}\\
&\mu_0(X_0) = 1 && \text{(1c)}\\
&\mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(1d)}\\
&\mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$.

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Recovery Algorithm

Relax LP to finite-dimensional LMI

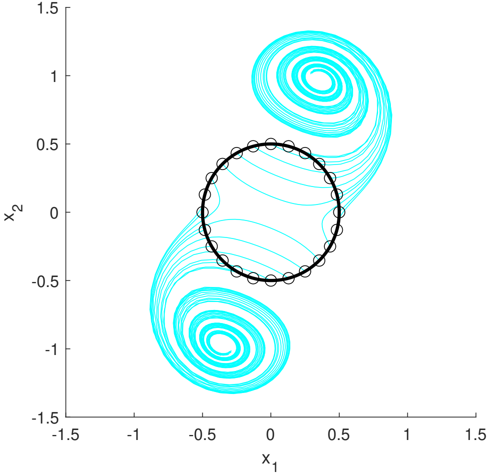
X compact, bounds $p_d^* \geq p_{d+1}^* \geq \ldots \geq p^*$

Attempt recovery if LMI solution has low rank

Related to optima extraction in polynomial optimization

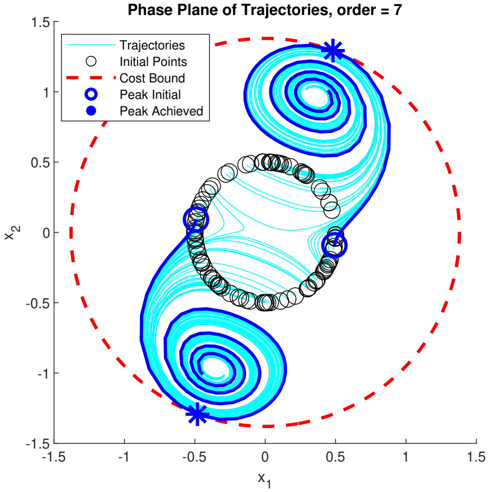
Compare recovered $p(x_p),\ p(x(t_p \mid x_0))$ with p_d^*

Symmetric Attractor Samples



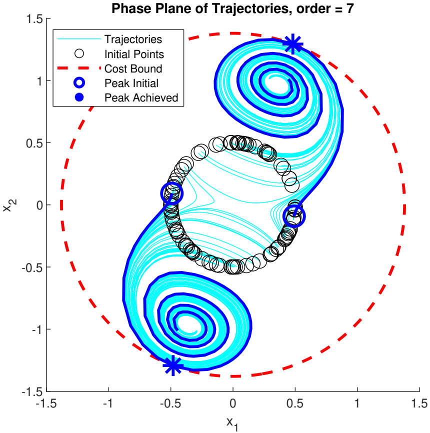
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.2x_1 + x_2 - x_2(x_1^2 + x_2^2) \\ -0.4x_1 + x_1(x_1^2 + x_2^2) \end{bmatrix}$$

Symmetric Attractor Norm Peak

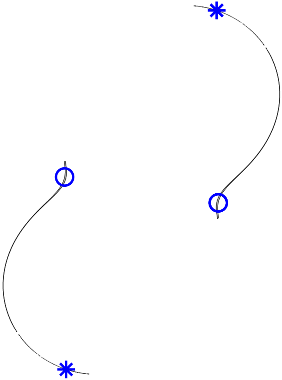


$p_7^* = 1.903 \geq \max \|x\|_2^2$, rank-2 initial/peak moment matrices

Sublevel Set Comparision



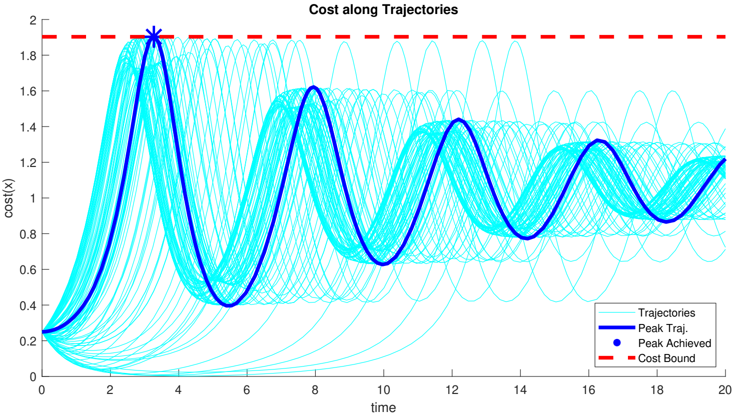
(a) Trajectories and Bounds



(b) near-optimal set

Successful Extraction vs. Sublevel Set Localization

Symmetric Attractor Norm Cost

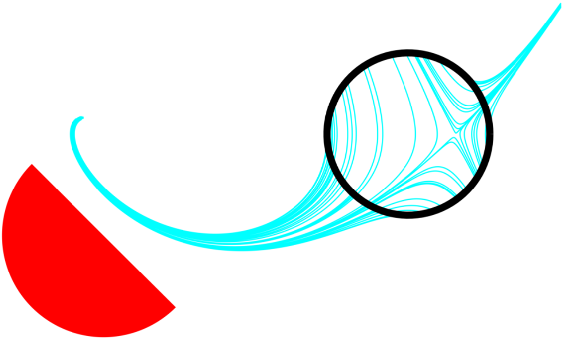


$\|x\|_2^2$ along trajectories

Peaks achieved at $t_p^* = 3.27$

Safety

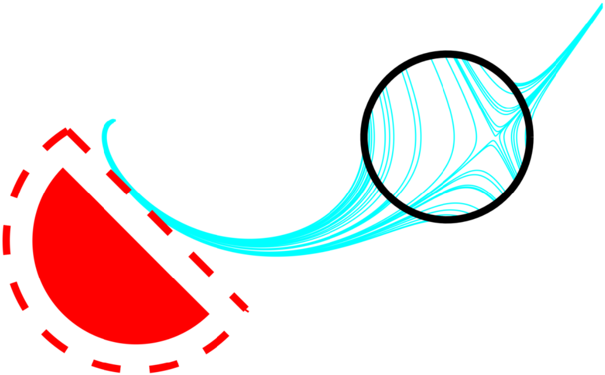
Safety Example System



Half-circle unsafe set

$$X_u = \{x \mid x_1^2 + (x_2 + 0.5)^2 \leq 0.5^2,\ x_1 + x_2 + 0.5 \leq 0\}$$

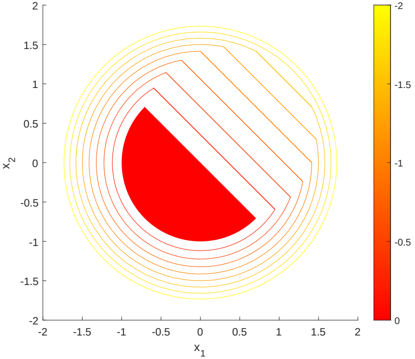
Safety Example Bounds



Trajectories are bounded away from X_u

Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$q \leq 1 - x_1^2 - x_2^2$ $\qquad$ $q \leq -x_1 - x_2$

$$q = -0.25, -0.5, \ldots, -2$$

Unsafe Sets

Unsafe set X_u:

$$\begin{aligned} X_u &= \{x \mid p_i(x) \geq 0 \; \forall i = 1 \ldots N_u\} \\ &= \{x \mid \min_i p_i(x) \geq 0\} \end{aligned}$$

Does a trajectory with $x(0) \in X_0$ enter X_u?

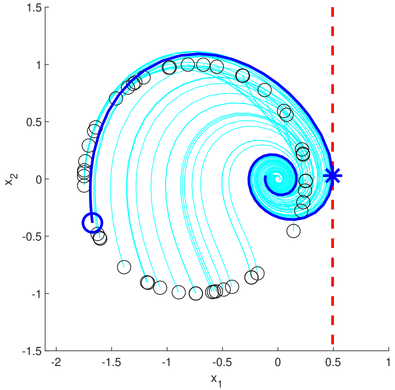
Barrier/Feasibility alternatives: binary determination

Maximin Objective

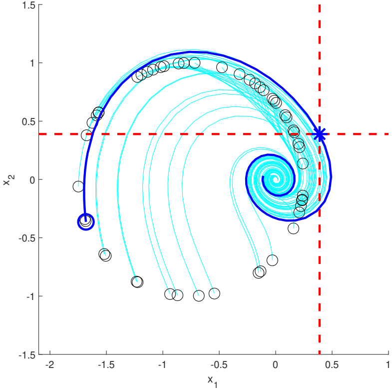
Maximize the minimum of functions $\{p_i(x)\}_{i=1}^{N_u}$

$$
\begin{aligned}
p^* = \max_{q \in \mathbb{R}} \quad & q \\
& q \leq \langle p_i(x), \mu_p \rangle & \forall i = 1 \ldots N_u \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle & \forall v \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Maximin Example



(a) $\max x_1$



(b) $\max \min(x_1, x_2)$

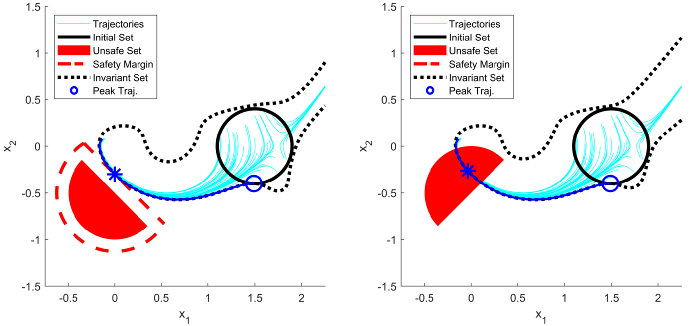
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 t - 0.1x_1 - x_1 x_2 \\ x_1 t - x_2 + x_1^2 \end{bmatrix}$$

Safety margin

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \; \forall i = 1 \dots N_u\}$

Safety margin $p^* = \max \min_i p_i(x)$ along trajectories

If $p^* \leq p_d^* \leq 0$, no trajectories enter X_u (safe)



(a) Safe: $p_5^* = -0.1417 < 0$ **(b)** Unsafe: $p_5^* = 0.1935 > 0$

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The Audience

Code: `http://github.com/jarmill/peak`