Assessing the Quality of a Set of Basis Functions for Inverse Optimal Control via Projection onto Global Minimizers

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Main Points

Background

Problem description

What we did

Take aways

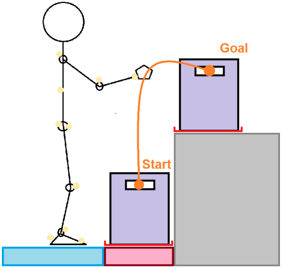
Background

Motivating Example 1

Context: Robotics, Biomechanics, Ergonomics

Box-Lifting Problem

$$\begin{aligned} \underset{\textbf{Joint trajectory}}{\text{minimize}} \quad & f(\theta, \textbf{Joint trajectory}) \\ \text{subject to} \quad & \textbf{Dynamics} \\ & \textbf{Joint limits} \\ & \textbf{Torque limits} \\ & \textbf{Collision avoidance} \\ & \textbf{Stability} \\ & \textbf{Boundary conditions} \end{aligned}$$

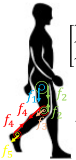


Motivating Example 2

Context: Biomechanics, Rehabilitation

Force-Sharing Problem

$$\begin{aligned} \underset{\textbf{Muscle forces}}{\text{minimize}} \quad & f(\theta, \textbf{Muscle forces}) \\ \text{subject to} \quad & \textbf{Force limits} \\ & \textbf{Consistent joint moments} \end{aligned}$$



$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \mathbf{Af} = \begin{bmatrix} r_{11} & -r_{12} & 0 & 0 & 0 \\ 0 & r_{22} & r_{23} & r_{24} & 0 \\ 0 & 0 & 0 & -r_{34} & -r_{35} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

Joint torques | Muscle moment-arms matrix | Muscle forces

Inverse Optimization

$$x^* \in \arg\min_{x \in X} f(x)$$

Type	Given	Desired
Direct Optimization	$f(\cdot),\ X$	x^*
Constraint Discovery	$f(\cdot),\ x^*$	X
Inverse Optimization	$x^*,\ X$	$f(\cdot)$

Convex combination of basis

Assumption 1: with basis $\mathcal{F} = \{f_j\}_{j=1}^m$ of (convex, C^1) cost functions

$$f_\alpha(x) = \sum_{j=1}^{m} \alpha_j f_j(x), \qquad \alpha \in \Delta^m$$

Problem 1: Feasibility

$$\underset{\alpha \in \Delta^m}{\text{find}} \; y \in \arg\min_{x \in X} f_\alpha(x)$$

Problem 2: Distance

$$p^* = \min_{x \in X} \|y - x\|_2^2$$
$$\exists \alpha \in \Delta^m \mid x \in \arg\min_{x' \in X} f_\alpha(x')$$

Bounding distance

Is basis $\mathcal{F} = \{f_j\}_{j=1}^m$ not good enough?

Option 1: Run global optimization on the non-convex bilevel distance problem.

Option 2: Find the lower bound on the distance using convex SDP.

Allows only to discard basis sets, not validate them.

Unconstrained

Unconstrained setting

Basis functions f_j are strongly convex and C^1.

Ground set $(x, \alpha) \in \mathcal{W} = \mathbb{R}^n \times \Delta^m$

Optima Sets

$$\hat{\mathcal{G}} = \{(x, \alpha) \in \mathcal{W} \mid \nabla_x f_\alpha(x) = 0\}$$
$$\mathcal{G} = \pi^x \hat{\mathcal{G}}$$

Projection Problem

$$p^* = \min_{(x,\alpha)\in\hat{\mathcal{G}}} \|y - x\|_2^2$$

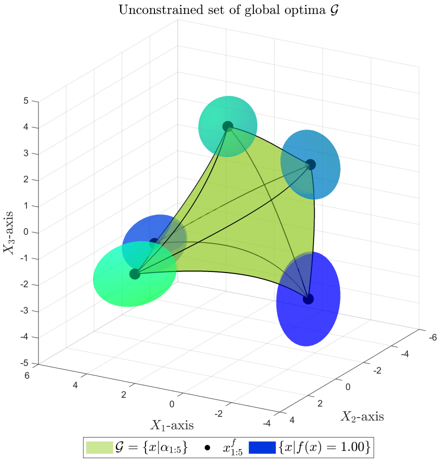
Mapping representation (continuous surjection)

$$\kappa : \Delta^m \mapsto \mathcal{G} \qquad \kappa(\alpha) = x = \arg\min_{x' \in \mathbb{R}^n} f_\alpha(x')$$

$\mathcal{G}$ quadratic example

$\mathcal{G}$ is compact and path-connected.

$$f_j(x) = \frac{1}{2}(x - x_j)^T Q_j (x - x_j), \qquad Q_j \in \mathbb{S}_{++}$$



Constrained

Constrained Setting

Each f_j weakly convex.

$$X = \{x \in \mathbb{R}^n \mid A_{eq}x = b_{eq},\ g_k(x) \leq 0\ \forall k = 1..r\}$$

Assuming Slater's condition holds

$$\exists x' \in \mathbb{R}^n \mid A_{eq}x' = b,\ g_k(x') < 0\ \forall k = 1..r$$

KKT Conditions

Necessary and sufficient for optimality

$$x = \arg\min_{x' \in \mathbb{R}^n} f_\alpha(x')$$

$$\iff$$

$$\exists \lambda \in \mathbb{R}^p,\ \exists \mu \in \mathbb{R}^r :$$

$$\nabla_x f_\alpha(x) + A_{eq}^T \lambda + \sum_{k=1}^{r} \mu_k \nabla_x g_k(x) = 0$$

$$A_{eq} x = b$$

$$g_k(x) \leq 0,\ \mu_k \geq 0 \qquad \forall k = 1..n$$

$$\sum_k \mu_k g_k(x) = 0.$$

Optimal Sets (Constrained)

Ground set

$$\mathcal{W}^c = \mathbb{R}^n \times \Delta^m \times \mathbb{R}^r_+ \times \mathbb{R}^q$$

Constrained-optimal sets

$$\hat{\mathcal{G}}^c = \{(x, \alpha, \mu, \lambda) \in \mathcal{W}^c \mid \text{KKT conditions (prev.) hold}\}$$
$$\mathcal{G}^c = \pi^x \hat{\mathcal{G}}^c$$

Projection Problem

$$p^* = \min_{(x,\alpha,\mu,\lambda)\in\hat{\mathcal{G}}^c} \|y - x\|_2^2.$$

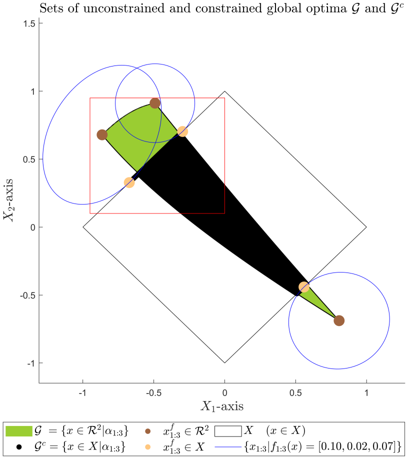
Mapping representation

$$\kappa^c : \Delta^m \rightrightarrows \mathcal{G}^c \qquad \kappa^c(\alpha) = \{x \in X \mid x = \arg\min_{x' \in X} f_\alpha(x')\}$$

$\mathcal{G}^c$ quadratic example

$$f_j(x) = \frac{1}{2}(x - x_j)^T Q_j (x - x_j), \qquad Q_j \in \mathbb{S}_{++}$$

$$g_k(x) = a_k^T x - b_k$$



Solving

Upper Bounds

Upper bounds: local search

Unconstrained (continuous)

$$F(\alpha) = \|y - (\sum_{j=1}^{m} \alpha_j Q_j)^{-1}(\sum_{j=1}^{m} \alpha_j \varphi_j)\|_2^2$$

Constrained (discontinuous)

$$\kappa^c(\alpha) = \arg\min_{x \in X} \sum_{j=1}^{n} x^T \alpha_j Q_j x/2 + \alpha_j \varphi_j^T x$$
$$F^c(\alpha) = \min_{x \in \kappa^c(\alpha)} \|y - x\|_2^2$$

monitor $\mathcal{N}(\sum_{j=1}^{m} \alpha_j Q_j)$

Unconstrained Lower Bounds

$$(x, \alpha) \in \mathcal{W} \quad \mapsto \quad M = [1\ x\ \alpha][1\ x\ \alpha]^T \in \mathbb{S}_+^{1+n+m}$$

$$M = \begin{bmatrix} M_{11} & M_{1x} & M_{1\alpha} \\ M_{x1} & M_{xx} & M_{x\alpha} \\ M_{\alpha 1} & M_{\alpha x} & M_{\alpha\alpha} \end{bmatrix}$$

$$\|y - x\|_2^2 \quad \mapsto \quad \sum_{i=1}^{n} (M_{x_i x_i} - 2 y_i M_{1 x_i}) + \|y\|_2^2$$

$$\alpha \in \Delta^m \quad \mapsto \quad \begin{cases} \sum_{j=1}^m M_{\alpha_j 1} = 1 \\ M_{\alpha_j 1} \geq 0 \qquad \forall j = 1..m \end{cases}$$

$$\nabla_x f_\alpha(x) \quad \mapsto \quad \sum_{j=1}^m Q_j M_{x \alpha_j} - (Q_j x_j^f) M_{1 \alpha_j} = 0$$

Valid inequalities

$$
\begin{aligned}
M_{\alpha_i 1} &= \textstyle\sum_{j=1}^{m} M_{\alpha_i \alpha_j} && \forall i = 1..m \\
M_{\alpha_i \alpha_j} &\geq 0 && \forall i \neq j \\
M_{\alpha_i \alpha_j} &\leq \alpha_i,\ M_{\alpha_i \alpha_j} \leq \alpha_j && \forall i \neq j \\
M_{\alpha_i \alpha_i} &\leq M_{1 \alpha_i} && \forall i = 1..m \\
M_{\alpha_i \alpha_j} &\leq 1/4 && \forall i \neq j.
\end{aligned}
$$

Constrained Lower Bounds

$$(x, \alpha, \lambda, \mu) \in \mathcal{W}^c \quad \mapsto \quad M = [1\; x\; \alpha\; \mu][1\; x\; \alpha\; \mu]^T \in \mathbb{S}^{1+n+m+r}$$

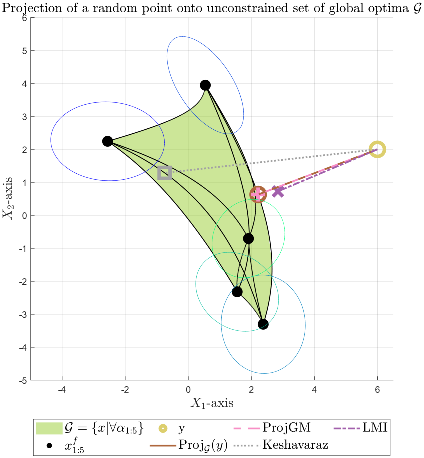
but affine λ does not enter the moment matrix.

Express KKT conditions through the moment matrix.

Add additional valid inequalities.

Projection unconstrained example

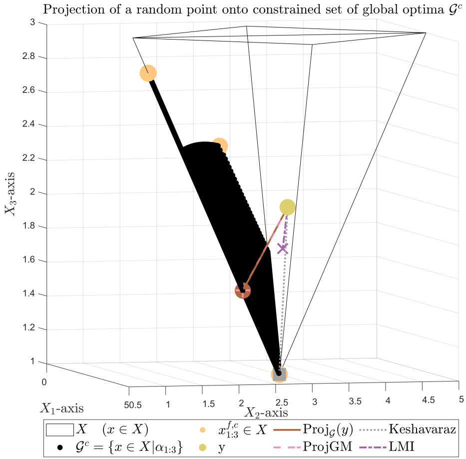
$$f_j(x) = \frac{1}{2}(x - x_j)^T Q_j (x - x_j), \qquad Q_j \in \mathbb{S}_{++}$$



Projection constrained example

$$f_j(x) = \frac{1}{2}(x - x_j)^T Q_j (x - x_j), \qquad Q_j \in \mathbb{S}_{++}$$

$$g_k(x) = a_k^T x - b_k$$



Extensions

Max-Representable Functions

Let $f_j(x) = \max_{\ell \in 1..L_j} w_{j\ell}(x)$ for L_j finite, $w_{j\ell}$ convex.

Convexity preserved over (pointwise) maxima

Cast unconstrained projection as lifted constrained problem

$$f^* = \min_{x \in X} \sum_{j=1}^{m} \alpha_j f_j(x)$$

$$\begin{aligned} f^* = \min_{x \in X,\ \tau \in \mathbb{R}^m} & \sum_{j=1}^{m} \tau_j \\ \alpha_j w_{j\ell}(x) \le \tau_j & \qquad \forall \ell = 1..L_j,\ j = 1..m. \end{aligned}$$

Can now handle piecewise-affine costs

Projection onto Local Minima

Nonconvex functions $f_j(x)$

Distance between y and a local minimum of $f_\alpha(x)$?

Unconstrained: $\nabla_x^2 f_\alpha(x) \succ 0$

Constrained: harder, Hessian form positive over critical cone

Take-aways

Conclusion

Minimize distance to global optima

Solve: local search, LMI + valid constraints

Handle max-representable piecewise functions

Future Work

- Analyze continuity properties
- Perform POP
- Project onto local minima
- Discover good cost candidates f_j

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Thank you for your attention

http://github.com/jarmill/inverse_optimal